where D_1 and D_2 are the characteristic functions (2) of the two systems and $p_1 = p(a)/p(b)$. A sufficient condition, then, that the characteristic numbers of the two systems shall either alternate or coincide, is that the quadratic form in $u_1(x)$, $u_2(x)$ shall be definite. But the discriminant Δ of the form is

$$\Delta = (p_1 - 1)^2 u_1^2.$$

Consequently the form will be definite if and only if we have $p(a) = p(b)$, which is the well known condition that system II shall be self-adjoint.*

THE UNIVERSITY OF WISCONSIN

NOTE CONCERNING THE ROOTS OF AN EQUATION

BY K. P. WILLIAMS

Professors Carmichael and Mason have published the following theorem.†

All roots of the equation

$$(1) \qquad x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$$

are, in absolute value, less than

$$(2) \qquad \sqrt{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}.$$

It is apparent that this limit may be greatly in excess of the actual maximum of the absolute values of the roots. An illustration of this fact is furnished by the equation

$$(3) \qquad x^n + x^{n-1} + \cdots + x + 1 = 0.$$

The theorem asserts that $\sqrt{n+1}$ is greater than the absolute value of any root. If n is large this is rather meager and inexact information, since all roots are in absolute value exactly 1, irrespective of the value of n.

* *Note on the roots of algebraic equations*, this BULLETIN, vol. 21 (1914), p. 21.

† This example is treated by Bôcher by different means in his *Leçons sur les Méthodes de Sturm*, 1917, pp. 83–91.

It is possible to modify in a very simple way the work of the authors quoted, and to obtain a theorem that in some cases gives much more exact knowledge. The work will merely be sketched in order to avoid unnecessary repetition of the work of the paper in question. It is shown in that paper that all roots of the given equation are not greater in absolute value than

$$\limsup_{m \to \infty} \sqrt[m]{C_m},$$

where

$$C_m = - \begin{vmatrix} 1 & 0 & 0 & \cdots & a_1 \\ a_1 & 1 & 0 & \cdots & a_2 \\ a_2 & a_1 & 1 & \cdots & a_3 \\ a_3 & a_2 & a_1 & \cdots & a_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

The theorem cited is then derived by applying to this determinant the theorem of Hadamard relative to a maximum value for a given determinant.

Before applying the theorem of Hadamard it is evidently possible to modify the determinant in various ways, and a new theorem will result from each modification. Let us subtract the first column from the last, the second from the first, the third from the second, etc. We have then the determinant

$$C_m = - \begin{vmatrix} 1 & 0 & \cdots & a_1 - 1 \\ a_1 - 1 & 1 & \cdots & a_2 - a_1 \\ a_2 - a_1 & a_1 - 1 & \cdots & a_3 - a_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n - a_{n-1} & \cdot & \cdots & - a_n \\ - a_n & a_n - a_{n-1} & \cdots & \\ 0 & - a_n & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \end{vmatrix}$$

assuming $m > n$, the determinant being of order m. The application of the theorem of Hadamard then gives the theorem:

THEOREM. *All roots of the equation* (1) *are less in absolute* *value than the quantity*

$$(4)\qquad \sqrt{1+|a_1-1|^2+|a_2-a_1|^2+\cdots+|a_n-a_{n-1}|^2+a_n^2}.$$

If we apply this theorem to equation (3) we see that all roots are less in absolute value than $\sqrt{2}$, irrespective of the value of n.

The quantity in (4) will evidently be smaller than the quantity (2) in many cases. As another illustration, consider the equation

$$(5)\qquad x^n+\frac{x^{n-1}}{\sqrt{1}}+\frac{x^{n-2}}{\sqrt{2}}+\frac{x^{n-3}}{\sqrt{3}}+\cdots+\frac{x}{\sqrt{n-1}}+\frac{1}{\sqrt{n}}=0.$$

The application of (2) gives

$$\sqrt{1+1+\frac{1}{2}+\cdots+\frac{1}{n}},$$

as a superior limit for the absolute values of the roots. Now Euler's constant tells us that this is of practically the same order as $\sqrt{\log n}$. We therefore could draw no conclusion as to whether the roots of (5) remain within a certain circle which does not change with n.

If, on the other hand, we apply (4), we have

$$\sqrt{1+\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)^2+\cdots+\left(\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}\right)^2+\frac{1}{n}},$$

as the superior limit. The quantity under the radical is less than

$$\begin{aligned}1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n}\\ =1+\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots\\ +\left(\frac{1}{n-1}-\frac{1}{n}\right)+\frac{1}{n}=2.\end{aligned}$$

We therefore see that no root of (5) will, in absolute value, exceed $\sqrt{2}$, no matter how great n may be.

INDIANA UNIVERSITY