SHORTER NOTICES

Irrationalzahlen. By Oskar Perron. Berlin und Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. viii + 186 pp. Volume I of Göschen's Lehrbücherei, Gruppe I, Reine Mathematik.

Göschen's *Lehrbücherei* is to comprise works of textbook character, on topics chosen from the fields of mathematics, the exact sciences, and "Technik." They are intended primarily for students of universities and of *Technische Hochschulen.* Each volume is assumed to cover approximately the ground of a one semester university course.

The volume under consideration contains two parts of essentially different character. While the first half may be said to give a systematic treatment of the notion of an irrational number and its historical development, the last chapters are devoted to the interesting and not very widely known subject of the various methods of representing irrational numbers and their approximation by rational numbers and to a brief discussion of a certain class of transcendental numbers.

These last chapters, while not containing a large amount of new material, offer much that may be of interest even to professional mathematicians, particularly since the problem of approximation of irrational numbers is steadily gaining in importance and represents a difficult, but fascinating, field for research in which German and English mathematicians are making remarkable discoveries.

The book is divided into six chapters:

I. *The foundations* (32 pp.): Assuming the existence of rational numbers, the irrational numbers are defined by the Dedekind *Schnitt.* Refined questions of independence and consistency of the axioms employed are, in agreement with the character of the book, not dealt with.

II. *The notion of a limit* (28 pp.): The notions *upper and lower bound, limited, upper and lower limit* of a set of numbers are introduced. The *limit,* $\lim_{n=\infty} a_n$, is introduced only after the $\limsup_{n=\infty} a_n$, $\liminf_{n=\infty} a_n$ have been, in six pages, thoroughly established, and is defined by $\liminf a_n = \limsup a_n = \lim a_n$.—Convergent sequences and infinite series are briefly considered. The chapter ends with a four-page historical review of the various established methods of introducing irrational numbers (Cauchy, Bolzano, Weierstrass, Dedekind, Cantor, Méray, Bachmann, etc.) and a brief but clear discussion of their relations to one another.

III. *Powers and logarithms* (30 pp.): This chapter introduces powers with rational and irrational exponents, logarithms, the exponential series, etc., in the customary manner. It may be noted that the author, for formal reasons, assigns to 0^0 the value 1.

IV. *Various forms of representation of irrational numbers* (36 pp.): The following representations are explained and discussed (convergence, uniqueness, exceptional cases, etc.):

(*a*) $\gamma = \sum_{\nu=0}^{\infty} c_\nu/p^\nu$, p an arbitrary positive integer, c_ν integers, $0 \leqq c^\nu \leqq p-1$ for $\nu \geqq 1$, $c_\nu \leqq p-2$ for an infinite number of ν's (*systematic fractions*).

(*b*) $\gamma = [b_0, b_1, b_2, \cdots]$, b_ν positive integers (*continued fractions*); periodicity, Lagrange's theorem, Lambert's fraction for $(e+1)/(e-1)$, Euler's fraction for e, etc.
(*c*) $\gamma = c_0 + \sum_{\nu=1}^{\infty} c_\nu/(p_1, p_2, \cdots, p_\nu)$, p_ν arbitrary positive integers, c_ν integers, $0 \leqq c_\nu \leqq p_\nu - 1$ for $\nu \geqq 1$, $c_\nu \leqq p_\nu - 2$ for an infinite number of ν's (*Cantor's series and Cantor's algorithm*). Irrational when infinite number of terms.

$$(d) \qquad \gamma = c + \frac{1}{q_1} + \sum_{\nu=1}^{\infty} \frac{1}{(q_1 - 1)q_1(q_2 - 1)q_2 \cdots (q_\nu - 1)q_\nu} \cdot \frac{1}{q_{\nu+1}},$$

c and q_ν integers, $q_\nu \geqq 2$ for $\nu \geqq 1$ (*Lüroth's series*). Irrational except when $q_1, q_2, q_3, \cdots$ form a periodic sequence.
(*e*) $\gamma = c + \sum_{\nu=0}^{\infty} (q_1 q_2 \cdots q_{\nu+1})^{-1}$, c, q_ν integers, $q_1 \geqq 2$, $q_{\nu+1} \geqq q_\nu$ (*Engel's series of the first kind*). Irrational except when, from a certain ν on, always $q_{\nu+1} = q_\nu$.
(*f*) $\gamma = c + \sum_{\nu=0}^{\infty} (q_{\nu+1})^{-1}$, c, q_ν integers, $q_1 \geqq 2$, $q_{\nu+1} \geqq (q_\nu - 1)q_\nu + 1$ (*Engel's series of the second kind*). Irrational except when for all sufficiently large ν, $q_{\nu+1} = (q_\nu - 1)q_\nu + 1$.

(*g*) $\gamma = \prod_{\nu=0}^{\infty}\left(1 + \frac{1}{q_\nu}\right)$, q_ν positive integers, $q_{\nu+1} \geqq q_\nu^2$ and not all $q_\nu = 1$

(*Cantor's products*). Irrational except when from a certain ν on $q_{\nu+1} = q^2$. Cantor's products are based on the identity

$$(1 - x) \cdot \prod_{\nu=0}^{n-1} (1 + x^{(2^\nu)}) = 1 - x^{(2^n)}.$$

The reviewer does not know of any other place where these different methods of representing irrational numbers have been collected and discussed.

V. *Approximation to irrational numbers by means of rational numbers* (30 pp.): This chapter offers a satisfactory elementary introduction to the theory of *rational* and *diophantine approximations*, which is constantly growing in importance. Starting with the well known approximation to an irrational number ξ, $|\xi - p/q| < q^{-2}$ *for an infinite number of fractions* p/q, and from Hurwitz' complementary theorems on this inequality, the author advances to the proof and discussion of the following theorem concerning the simultaneous approximation of any finite number of given numbers $\xi_1, \cdots, \xi_n$: $|\xi_\nu - p_\nu/q| < q^{-1-1/n}$ *for an infinite number of* p_ν/q, *provided the* ξ_ν *are not all rational.*

After considering more complicated systems, a deep-lying theorem of Kronecker* concerning the simultaneous approximation of a system of linear functions of any number of variables is derived.

The author claims a considerable simplification as compared with Kronecker's proof. The Kronecker theorem has found within the last

* Kronecker, *Näherungsweise ganzzahlige Auflösung linearer Gleichungen*, SITZUNGSBERICHTE DER PREUSSISCHEN AKADEMIE ZU BERLIN, 1884, p. 1179 ff. The complete theorem is too long and complicated to be quoted here.

years unexpected applications, for example in the theory of Riemann's Zeta-function.

VI. *Algebraic and transcendental numbers* (24 pp.): Besides the fundamental definitions and theorems (including the proof that e and π are transcendental numbers) a certain class of transcendental numbers which have been called by Maillet* *Liouville numbers* are studied. Since Maillet introduced the name and made in his book a systematic (although somewhat obscure) study of these numbers and since Perron in another work (*Die Lehre von den Kettenbrüchen*, Leipzig, 1913) gives full credit to Maillet, it is obviously an oversight that Maillet's book is not mentioned in the *Irrationalzahlen*.

The literature references are arranged for each chapter separately and seem fairly complete. However, Minkowski, *Diophantische Approximationen*, Leipzig, 1907, is not quoted. Borel, *Leçons sur la Théorie de la Croissance*, Paris, 1910, pp. 118–168, might have been mentioned in connection with chapters V, VI. It is not very satisfactory that even in the case of large books no page reference is given; a reference such as: L. Euler, *Introductio in Analysin infinitorum*, I, 1748 (a book of over 300 quarto pages, in Latin), is perhaps not easily run down.

The only American author referred to seems to be Huntington (TRANSACTIONS OF THIS SOCIETY, vol. 6 (1905).

A. J. KEMPNER

Grundlehren der Neueren Zahlentheorie. By Paul Bachmann. Volume III of Göschen's Lehrbücherei, Gruppe I, Reine Mathematik. Berlin und Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. xi + 252 pp.

The book under discussion represents the second edition of volume LIII of the well known Sammlung Schubert, G. J. Göschensche Verlagshandlung, Leipzig, 1907. No important changes have been made. New is a chronological table of the known proofs of the famous *Law of Quadratic Reciprocity:* fifty-six proofs from the year 1796 (Gauss' first proof, published 1801) to Frobenius' modification of Zeller's proof, 1914.† The mathematical basis of each proof is indicated; in thirty-two cases Gauss' lemma or a variant of Gauss' lemma is given as the foundation. A five-page alphabetic index has also been added.

Since the first edition was reviewed by J. W. Young in this BULLETIN (vol. 15 (1908–9), pp. 463-5), it is not necessary to consider in detail the mathematical contents of this excellent little book. The small corrections which were suggested in this review have been carried out.

The new edition is posthumous; it contains a five-page necrology, *Zum Gedächtnisse von Paul Bachmann*, by Robert Haussner. Bachmann's influence in stimulating interest in the theory of numbers has been so great that American readers may be interested in a few notes concerning his life and his work.

---

* Maillet, *Introduction à la Théorie des Nombres Transcendants*, Paris, 1906, particularly chapters II, III.

† A less complete table is contained in Bachmann's *Niedere Zahlentheorie*, vol. I, p. 203 ff.