SOME PROPERTIES OF CONTINUOUS CURVES*

BY G. T. WHYBURN

The points A and B of a continuum M are said to be separated in M by a point X of M if $M-X$ is the sum of two mutually separated sets S_1 and S_2 containing A and B respectively. The point P of a continuum M is a cut point of M if and only if the set of points $M-P$ is not connected, i.e., is the sum of two mutually separated point sets.

A continuous curve M will be said to be *cyclicly connected* provided that every two points of M lie together on some simple closed curve which is contained in M. In this paper use will be made of the following fundamental theorem.

THEOREM A. *In order that the continuous curve M should be cyclicly connected it is necessary and sufficient that M should have no cut point.*

A proof for Theorem A will be found in my paper *Cyclicly connected continuous curves*, which will appear soon.

THEOREM I. *If A and B are any two points of a continuous curve M and if K denotes the set of all those points of M which separate A from B in M, then $K+A+B$ is a closed set of points.*

PROOF. The curve M contains a simple continuous arc t from A to B. Clearly K must be a subset of t. Let P be any point of $t-(K+A+B)$. Since P does not belong to K, A and B must both belong to some connected subset† of $M-P$, and by a theorem of R. L. Moore's‡ it follows that $M-P$ contains an arc t' from A to B. On the arcs PA

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† See an abstract of a paper by R. L. Wilder, *A characterization of continuous curves by a property of their open subsets*, this Bulletin, vol. 32 (1926), pp. 217–218.

‡ *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 255.

and PB of t, in the order from P to A and from P to B respectively, let X and Y respectively denote the first points belonging to t'. Then no point of the segment XPY of t can belong to $K+A+B$. Since thus every point of $t-(K+A+B)$ belongs to some segment of t which contains no point of $K+A+B$, the set $K+A+B$ is closed.

DEFINITION. A cyclicly connected continuous curve C is said to be a *maximal cyclic curve* of a continuous curve M if C is a subset of M and is not a proper subset of any other cyclicly connected continuous curve belonging to M.

THEOREM II. *If A and B are any two points of a continuous curve M, t is any arc of M from A to B, K denotes the set of all those points of M which separate A from B in M, and S is any maximal segment of $t-(K+A+B)$, then M contains a maximal cyclic curve which contains S.*

PROOF. Let E and F denote the end points of S. Let G denote the collection of all the maximal connected subsets of $M-(E+F)$. At least one element of G, namely, the one which contains S, must have both of the points E and F for limit points. Only a finite number of elements of G can have this property. Let H denote the point set obtained by adding together all the elements of G which do have this property, and let N denote the point set $H+E+F$. Then N is closed and connected. The continuum N is also connected im kleinen. For since M is connected im kleinen, no point of H is a limit point of $M-H$. Hence H is connected im kleinen. Then from a theorem of R. L. Moore's* it easily follows that $H+E+F=N$ must also be connected im kleinen. Thus we see that N is a continuous curve.

Now if N has no cut point, then by Theorem A, N must be cyclicly connected. If N has any cut points, then for every cut point X of N, let H_x denote the maximal connected subset of $N-X$ which contains $(S+X)-X$, and let N_x denote the point set H_x+X. That H_x exists in case X is

* *A report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), pp. 296–297.

not on S is obvious. If X belongs to S, then since X does not belong to $K+A+B$, $M-X$ contains an arc* t' from A to B. The arc t' must contain the points E and F, for E and F belong to $K+A+B$. The arc EF of t' must belong to N, hence also to $N-X$. Therefore, $S-X$, being the sum of the segments EX and EF of t, must lie in some connected subset of $N-X$. Thus, in any case, H_x exists. Now let L denote the set of all points which are common to all the point sets N_x. Clearly L exists, is a closed point set, and contains S. Let C denote the maximal connected subset of L which contains S. I shall show that C is a maximal cyclic curve of M.

Clearly C is closed and connected. I shall first show that if P and Q are any two points of C and POQ is any arc of N from P to Q, then every point of POQ must belong to C. This must be true, for if X is any cut point of N not on POQ, then H_x contains every point of POQ because it contains P and Q. And if X is any cut point of N on POQ, then H_x contains $POQ-X$, because $POQ-X$ is either a single connected set containing a point of H_x or the sum of two connected sets each of which contains a point of H_x. Hence, in any case, POQ belongs to every set N_x and therefore belongs to L and to C.

The continuum C is a continuous curve. For let P be any point of C and ϵ any positive number. Then since N is connected im kleinen, there exists a positive number δ_ϵ such that every point of N whose distance from P is less than δ_ϵ can be joined in N to P by an arc which is of diameter less than ϵ. Let X be any point of C whose distance from P is less than δ_ϵ. Then N contains an arc a from X to P of diameter less than ϵ. But as was shown above, a must belong to C. Hence C is connected im kleinen at every one of its points and is therefore a continuous curve.

The curve C has no cut point. For suppose, on the contrary, that C has a cut point X. Then $C-X=S_1+S_2$,

* See R. L. Wilder, loc. cit., and R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.

where S_1 and S_2 are mutually separated point sets. Let P_1 and P_2 be points of S_1 and S_2 respectively. Now if X is a cut point of N, then since H_x contains P_1 and P_2, it follows* that $N-X$ contains an arc s from P_1 to P_2. And if X is not a cut point of N, again it follows that $N-X$ contains an arc s from P_1 to P_2. And in either case, as was shown above, the arc s must belong to C. Hence S_1 and S_2 are not mutually separated, contrary to supposition. Therefore C has no cut point, and by Theorem A it follows that C is a cyclicly connected continuous curve. That C is a maximal cyclic curve of M follows immediately from the facts (1) that every arc of M joining two points of N must belong wholly to N and (2) that every arc of N joining two points of C must belong wholly to C. This completes the proof.

THEOREM III. *If A and B are any two points of a continuous curve M and if K denotes the set of all those points of M which separate A from B in M, then M contains two simple continuous arcs t_1 and t_2 from A to B whose common part is $K+A+B$.*

PROOF. By Theorem I, $K+A+B$ is a closed set of points. Hence if t is any arc of M from A to B, t must contain K, and $t-(K+A+B)$ is the sum of a countable number of non-overlapping segments $S_1, S_2, S_3, \cdots$. By Theorem II it follows that for every positive integer i, M contains a maximal cyclic curve C_i which contains S_i. For each i, let the end points of S_i be denoted by A_i and B_i. Since C_i is cyclicly connected, then for each i, C_i contains two arcs t_{1i} and t_{2i} from A_i to B_i whose common part is only the points A_i and B_i. Let

$$t_1 = K + A + B + \sum_{i=1,2,3,\cdots} t_{1i}, \qquad t_2 = K + A + B + \sum_{i=1,2,3,\cdots} t_{2i}.$$

Then t_1 and t_2 are simple continuous arcs of M from A to B whose common part is $K+A+B$.

THE UNIVERSITY OF TEXAS

* See R. L. Moore, loc. cit.