Again, generally $f_1 \neq f'$. Under what condition will $f_1 = f'$? From (2) and (3) we see that *the dual of a function is the same as the negative of the function if each discriminant is the dual-negative of its conjugate.*

Finally, if we have a relation $f=0$, then in general $f_1 \neq 1$, though always $f'=1$. What is the condition that $f_1=1$ when $f=0$? By means of (2) and (3) we find this condition to be *the same as the condition that* $f_1=f'$.

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THE HEAVISIDE OPERATIONAL CALCULUS*

BY H. W. MARCH

In a number of recent papers, Carson† has made a definite advance in the study of the Heaviside operational calculus by showing that the solution of an operational equation of the type in question can be obtained from an integral equation. Having done this, he was able to discuss Heaviside's three principal rules and to derive a number of important theorems by the use of which it is possible to solve by operational methods, problems to which Heaviside's rules are not directly applicable.

Somewhat earlier Bromwich‡ and Wagner§ solved, by the use of contour integrals in the complex plane, problems to which one of Heaviside's rules is applicable. They noted that the corresponding rule of Heaviside, the expansion theorem, follows at once from a calculation of the residues at the poles of the integrand in the case of a suitably restricted

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† J. R. Carson, Bell Technical Journal, vol. 4 (1925), pp. 685–761; vol. 5 (1926), pp. 50–95, 336–384; this Bulletin, vol. 22 (1926), pp. 43–68. See also P. Levy, Bulletin des Sciences Mathématiques, vol. 50 (1926), pp. 174–192.

‡ T. J. I'A. Bromwich, Proceedings of the London Society, (2), vol. 15 (1916), pp. 401–448; see also H. S. Carslaw, Philosophical Magazine, vol. 39 (1920), pp. 603–611.

§ K. W. Wagner, Archiv für Elektrotechnik, vol. 4 (1916), pp. 159–193.

operator. In a later paper Bromwich* also obtained from a contour integral the asymptotic solution for a particular problem.

It is the purpose of the present paper to show the connection between the methods of Carson and Bromwich by showing that Bromwich's contour integral is the solution of the integral equation set up by Carson. In the opinion of the writer the two methods supplement one another in a valuable way. To emphasize the importance of the contour integral in obtaining Heaviside's rules rigorously and simply, the derivation of each of the three principal rules is briefly indicated.

Carson showed that the solution $h(t)$ of the operational equation†

$$h(t) = \frac{1}{H(p)} \tag{1}$$

which satisfies the appropriate initial conditions is the solution of the integral equation

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt, \tag{2}$$

the real part of p being assumed to be negative.

Bromwich, on the other hand, attacked the problem directly without using the operational calculus, by assuming the solution of the corresponding differential equation or system of differential equations to be expressed by a contour integral in the complex plane. He found that the solution $h(t)$ is given in certain cases by the integral‡

$$h(t) = \frac{1}{2\pi i}\int_C \frac{e^{pt}}{pH(p)}dp, \tag{3}$$

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* Proceedings, Cambridge Philosophical Society, vol. 20 (1920–21), pp. 423–427.

† For notation and details, see Carson, Bell Technical Journal, vol. 4 (1925), pp. 686–705; this Bulletin, vol. 22 (1926), pp. 43–47.

‡ Proceedings of the London Society, (2), vol. 15 (1916), pp. 410–420.

or by the integral

$$(4)\qquad h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt\infty}}{pH(p)}dp,$$

where C in (3) is a closed curve surrounding all the poles of the function

$$\frac{1}{pH(p)},$$

no one of which has, by hypothesis, a positive real part, and where the path of integration in (4) is a straight line parallel to the axis of imaginaries, c being any positive real number. The path of integration in (3) results from a deformation of the path in (4).

It will now be shown that the solution* of the integral equation (2) is given by (4). Fourier's theorem can be written in the form†

$$(5)\qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xy}dy\int_0^\infty e^{-yz}f(z)dz,$$

where c and the path of integration are chosen as in (4). If we write

$$(6)\qquad g(y) = \int_0^\infty e^{-zy}f(z)dz,$$

it follows from (5) that

$$(7)\qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(y)e^{xy}dy.$$

Accordingly (7) furnishes the solution‡ of the integral equation (6) provided that $g(y)$ is subject to suitable restrictions. These must be such that (6) can be shown to be satisfied when $f(z)$ as given by (7) is substituted in it. This

* See the recent paper by J. D. Tamarkin, Transactions of this Society, vol. 28 (1926), p. 417.

† See, e.g., H. M. MacDonald, Proceedings of the London Society, vol. 35 (1902), p. 428. See also B. Riemann, *Gesammelte Werke*, p. 149.

‡ For a proof of the uniqueness of a continuous solution of equation (6), see M. Lerch, Acta Mathematica, vol. 27 (1903), pp. 339–351.

can be done if $g(y)$ is an analytic function of y at all points in the half-plane in which the real part of y is positive and if further, writing $y=\rho e^{i\theta}$, it is true that in this half-plane

$$| g(y) | < \frac{N}{\rho^\alpha},$$

for a sufficiently large ρ, N being a sufficiently large fixed positive number, and α being a fixed positive number. In accordance with (7) the function $h(t)$ defined by equation (4) in terms of Bromwich's contour integral is the solution of Carson's integral equation (2).

Heaviside's rules can be derived in a much more direct and rigorous manner from the contour integral than from the integral equation. Indeed the contour integral appears to furnish the key to the whole situation, making it possible to determine whether or not in a given case the Heaviside rule in question is applicable and to discuss in a satisfactory manner the results obtained by applying these rules. If it appears in a given case that Heaviside's rules are not applicable, the result is to be sought by studying the contour integral itself.

We shall now indicate briefly the derivation of Heaviside's three principal rules and give an example in which it can be seen from the contour integral why Heaviside's rule does not lead to a correct result.

(a) *The Power Series Solution.* Assume that the function $H(p)$ is such that the path of integration in (4) can be deformed into a circle C described about the origin as center, and that on C, $1/H(p)$ can be expanded in the convergent series

$$\frac{1}{H(p)} = a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n} + \cdots .$$

Also expand e^{pt} in a power series in p. Then

$$h(t) = \frac{1}{2\pi i}\int_C \left(\frac{a_0}{p} + \frac{a_1}{p^2} + \frac{a_2}{p^3}\cdots\right) \cdot\left(1 + pt + \frac{p^2t^2}{2!} + \frac{p^3t^3}{3!} + \cdots\right)dp.$$

On forming the product of these series term by term, and arranging according to negative and positive powers of p, we find the Laurent expansion of the integrand in a circular ring enclosing the origin as center. The value of the integral is $2\pi i$ times the coefficient of the term in $1/p$. Hence we have

$$h(t) = a_0 + a_1 t + \frac{a_2 t^2}{2!} + \cdots + \frac{a_n t^n}{n!} + \cdots.$$

(b) *The Expansion Theorem.** Assume that $H(p)$ is a rational function which vanishes for n distinct values of p, but which does not vanish for $p=0$, and that the path of integration can be deformed into a circle c described about the origin as center and enclosing all the roots of $H(p)=0$. On calculating the residues at the points $0, p_1, p_2, \cdots, p_n$, the poles of the integrand, it is found that

$$h(t) = \frac{1}{H(0)} + \sum_{j=1}^{j=n} \frac{e^{p_j t}}{p_j H'(p_j)}\cdot$$

The same treatment applies when $H(p)=0$ has multiple roots or when $p=0$ is a root. By an appropriate deformation of the path of integration it is possible to extend the method to cases in which $H(p)$ is a transcendental function.

(c) *The Asymptotic Solution.* The contour integral lends itself equally well to the discussion of the third of Heaviside's principal rules, the asymptotic solution, and the one for which Carson's treatment was not entirely satisfactory. The method given below of obtaining the asymptotic solution was developed independently by the writer but its essential features were discovered earlier by Bromwich† in treating a particular case.

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* The proof outlined here was published almost simultaneously by Bromwich and Wagner in 1916. For other interesting derivations of the expansion theorem, see Bromwich, Philosophical Magazine, vol. 37 (1919), pp. 407–19; and F. D. Murnaghan, this Bulletin, vol. 33 (1927), p. 81.

† Proceedings, Cambridge Philosophical Society, vol. 20 (1920–21), pp. 423–427.

Heaviside's rule states that if $1/H(p)$ can be expanded in the form

$$(8)\quad \frac{1}{H(p)} = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots$$
$$+ (b_0 + b_1 p + b_2 p^2 + \cdots + b_n p^n + \cdots)p^{1/2},$$

$h(t)$ is obtained by discarding all terms in the first line excepting a_0, by replacing $p^{1/2}$ in the second line by $1/(\pi t)^{1/2}$ and p^n in this line by d^n/dt^n. It results that

$$(9)\quad h(t) = a_0 + \left(b_0 + b_1\frac{d}{dt} + b_2\frac{d^2}{dt^2} + \cdots\right)\frac{1}{(\pi t)^{1/2}}$$
$$= a_0 + \frac{1}{(\pi t)^{1/2}}\left(b_0 - b_1\frac{1}{2t} + b_2\frac{1\cdot 3}{(2t)^2} - b_3\frac{1\cdot 3\cdot 5}{(2t)^3} + \cdots\right).$$

Assume that $H(p)$ is such a function that the path of integration of (4) can be deformed into the path $ANPQB$ (see Fig. 1) enclosing the origin. This can be done in all

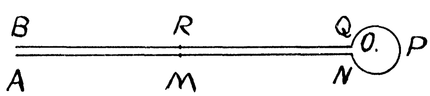

FIG. 1

cases which I have examined in which the formula can be applied and presumably in all such cases.

If $1/H(p)$ can be expanded in the form (8), we note that the integral of each term of the expression

$$\frac{1}{p}(a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots)e^{pt}$$

along the path $ANPQB$ is zero, while

$$\frac{1}{2\pi i}\int \frac{a_0}{p}e^{pt}dp = a_0,$$

the integral being taken along the same path as before. Accordingly, if we pass over, for the moment, all questions of convergence and make use of a proper determination of $p^{1/2}$ by writing $p^{1/2} = -i\rho^{1/2}$ along AN and $p^{1/2} = i\rho^{1/2}$ along QB, we find

$$
\begin{aligned}
h(t) &= a_0 + \frac{1}{2\pi i}\int_{AN} \frac{e^{pt}}{p^{1/2}}(b_0 + b_1 p + b_2 p^2 + \cdots)dp \\
&\quad + \frac{1}{2\pi i}\int_{QB} \frac{e^{pt}}{p^{1/2}}(b_0 + b_1 p + b_2 p^2 + \cdots)dp \\
&= a_0 + \frac{1}{\pi}\int_0^\infty (b_0 - b_1\rho + b_2\rho^2 - b_3\rho^3 + \cdots)\frac{e^{-\rho t}}{\rho^{1/2}}d\rho
\end{aligned} \tag{10}
$$

Equation (9) follows immediately on integrating term by term.

This analysis, which is purely formal, has led to the asymptotic solution and gives an intelligible foundation for this rather strange rule of Heaviside's. In certain cases the series in (9) is convergent for all positive values of t, in other cases it is only asymptotically convergent, and in others it is meaningless.* In any case, it is necessary to study the integral and the calculation outlined in obtaining equations (9) and (10). If the coefficients $b_0, b_1, b_2 \cdots$ are such that the series (9) is convergent for positive values of t, the investigation is usually not difficult. If the series is only asymptotically convergent, it will be found in many cases that we can proceed in the following manner. If possible, write the operator in the form

$$
\begin{aligned}
\frac{1}{pH(p)} = \frac{1}{p}&(a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n) \\
&+ (b_0 + b_1 p + b_2 p^2 + \cdots + b_n p^n + r_{n+1} p^{n+1}) p^{1/2},
\end{aligned}
$$

in which corresponding to a fixed n,

$$| r_{n+1} | < A,$$

provided that $|p| < k$, A being a fixed positive number. Let the points corresponding to $p = -k$ be denoted by M and R in Fig. 1. It can usually be shown that t can be chosen so large that the integrals of $e^{pt}/(pH(p))$ from $-\infty$ to M and

* Carson, Bell Technical Journal, vol. 4 (1925), pp. 744 and 749.

from R to $-\infty$, are arbitrarily small in absolute value. Also t can be chosen so large that the integrals from M to N and from Q to R of $r_{n+1}\, p^{n+1/2}\, e^{pt}$ is arbitrarily small. Then for a sufficiently large t, the absolute value of the difference between $h(t)$ and

$$a_0 + \frac{1}{\pi} \int_0^k [b_0 - b_1\rho + b_2\rho^2 - \cdots + b_n(-\rho)^n] \frac{e^{-\rho t}}{\rho^{1/2}} d\rho$$

is an arbitrarily small positive number. After this, the limit k can be replaced by ∞ for a sufficiently large t. Then $h(t)$ is given approximately by

$$h(t) = a_0 + \frac{1}{(\pi t)^{1/2}} \left[b_0 - \frac{b_1}{2t} + b^2 \frac{1 \cdot 3}{(2t)^2} - b_3 \frac{1 \cdot 3 \cdot 5}{(2t)^3} + \cdots + (-1)^n b_n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2t)^n} \right].$$

The importance of the contour integral in determining the applicability of Heaviside's rules is illustrated in the following example. Carson* showed that the application of the method of the asymptotic solution to the operational equation

$$h(t) = \frac{p^{1/2}}{p^2 + \omega^2},$$

led to an incorrect result. The reason that it does so is readily seen from a consideration of the contour integral

$$h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p^{1/2}}{p^2 + \omega^2} e^{pt} dp.$$

The path of integration cannot be deformed into the path of Fig. 1, because of the presence of singularities of the integrand at the points $p = \pm i\omega$. It is accordingly not to be expected that the asymptotic solution can be employed.

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* Bell Technical Journal, vol. 4 (1925), p. 749.