Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle. By Serge Bernstein. Paris, Gauthier-Villars, 1926. 10+207 pp.

This book, which is one of the Borel series of monographs, forms an exposition of the course of lectures given by Professor Bernstein at the Sorbonne in May, 1923. The results are for the most part due to the author himself, and the majority of them appear here in print for the first time. The work as a whole is a development of the methods of Tchebycheff in the study of the field indicated by the title. The great variety of interesting and important results that are here contained in some two hundred pages, furnishes the best possible support for the writer's contention that the ideas of Tchebycheff are of fundamental importance in this domain of the theory of analytic functions of a real variable.

The first chapter is concerned with extremal properties on a finite segment of polynomials and other functions depending on a given number of parameters, and develops the algebraic basis for the whole subject. It contains among other important theorems a classical result of the author concerning the best approximation to $|x|$ by polynomials of given degree. We also find here his fundamental theorem with regard to the relationship between the maximum modulus of a polynomial and that of its derivative.

In the second chapter a study is made of extremal properties on the entire axis of reals of algebraic functions and of certain types of integral functions. In connection with the application of these properties to polynomial approximation there is introduced the notion of a "function of comparison", $\phi(x)$, and the extremal properties of the expression

$$\epsilon_n = \left| \frac{f(x) - P_n(x)}{\phi(x)} \right|,$$

where f is a given function and P_n an arbitrary polynomial of degree n, are studied. It will be seen that this use of a function of comparison is analogous to the notion of relatively uniform convergence introduced by E. H. Moore. We find also in this chapter the interesting theorem that any continuous function of a real variable, $f(x)$, such that $\lim_{x\to+\infty} f(x) = A$, can be uniformly approached on the whole axis of reals by means of rational fractions possessing given poles $\alpha_n \pm \beta_n$, provided the series $\Sigma|\beta_n|/(\alpha_n^2+\beta_n^2)$ diverges. On the other hand, the uniform approach to $f(x)$ by rational fractions, possessing poles such that the above series is convergent, implies the fact that $f(x)$ is an analytic function having no singularities in the finite plane other than poles at $\alpha_n \pm \beta_n$. In the latter part of the chapter the fundamental results regarding the relationship between the maximum of the modulus of a polynomial and that of its derivative, referred to in connection with Chapter II, are generalized to the case of integral functions.

The third chapter deals with the question of the best approximation to analytic functions possessing given singularities. As that topic had previously been treated by de la Vallée Poussin in his *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, also of the Borel series, Professor Bernstein places the emphasis on those problems that were not discussed in the above mentioned monograph. He studies in particular the determination of the successive terms in the asymptotic development of the best approximation and the problem of the best approximation to a function having an essentially singular point. This latter is a problem of unusual difficulty, and it is noteworthy that in a wide variety of cases a solution of the problem can be obtained by the use of the same methods as in the case of polar singularities.

The book closes with two notes. The first one, which is rather extensive, develops certain interesting relationships between the problem of analytic extension of analytic functions of real variables and that of polynomial approximation to such functions. The second note, which is brief, deals with a property of analytic functions of genre zero.

As a whole the book has to a very great extent that admirable characteristic of the Borel series: namely, that its interest lies not only in the topics specifically treated, but also in the variety of problems for further research that are either definitely indicated or incidentally suggested.

C. N. MOORE

Potentialtheorie. By Dr. W. Sternberg. Two volumes. I. *Die Elemente der Potentialtheorie,* 136 pages, 1925. II. *Die Randwertaufgaben der Potentialtheorie,* 133 pages, 1926. Volumes 901 and 944 of the Sammlung Göschen. Berlin, Walter de Gruyter and Co.

The two volumes give a brief introduction to the theory of the various potentials in two and three dimensional space and to the two boundary value problems. The theorems and methods are in general well known. The author has attempted to reduce the necessary preparation of the reader to a minimum. A knowledge of the calculus is sufficient except in the case of the chapter on the Fredholm equation. After a short *mengentheoretische* introduction, the first volume takes up the definition and some properties of volume, surface, and double layer potentials. It is shown that the definition of the potential as a solution of Laplace's equation is coextensive with the usual integral definition. This new definition necessitates a new definition of mass and in this the author follows Plemelj. After a chapter on the necessary variations of Stokes' Theorem, the volume closes with a discussion of the continuity properties of the potentials and their derivatives. In the first part of the second volume the author confines himself chiefly to a statement of the Dirichlet and Neumann problems, their solution for the circle and sphere and a discussion of the Green's function. No mention is made of the method of Neumann or of the method of *balayage* of Poincaré. The remainder of the volume is given over to the theory of Fredholm's equation and its application to the boundary value problems. The chapter on the Fredholm equation is rather too brief to be useful to one who knows nothing of the theory. The work is carefully written and though very concise, is readable and is to be recommended.

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