I shall close with one further observation as to the properties of the set M. Mazurkiewicz calls* a set M quasi-connected if for every point m of M there corresponds a positive number λ such that there does not exist any division of M into two mutually separated sets M_1 and M_2 such that M_1 contains m and the diameter of M_1 is less than λ. He gives† an example of a quasi-connected set which contains no true quasi-components. That the set M constructed above is another example of such a set is easily shown; indeed, the value of λ may be taken *uniformly* equal to unity for all points of M.

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A SIMPLE METHOD FOR NORMALIZING TCHEBYCHEFF POLYNOMIALS AND EVALUATING THE ELEMENTS OF THE ALLIED CONTINUED FRACTIONS‡

BY J. A. SHOHAT [J. CHOKHATE]

1. *Introduction.* Consider a system

$$(1) \qquad P_n(x), \qquad (n = 0, 1, 2, \cdots),$$

of orthogonal, but not normal, Tchebycheff polynomials corresponding to a given (finite or infinite) interval (a, b) with the characteristic function $p(x)$. The corresponding *normalized* system of polynomials will be denoted by

$$(2) \qquad \phi_n(p\,;\,x) \equiv \phi_n(x) = a_n(p)\left[x^n - S_n(p)x^{n-1} + \cdots\right], \qquad (n = 0, 1, \cdots, a_n > 0).$$

We have, then,

$$(3) \qquad \int_a^b p(x)\phi_m(x)\phi_n(x)dx = \begin{cases} 0, m \neq n, \\ 1, m = n. \end{cases}$$

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* *Sur les ensembles quasi-connexes,* Fundamenta Mathematicae, vol. 2 (1921), pp. 201–205.

† Loc. cit.

‡ Presented to the Society, April 16, 1927.

The normalization of the system (1), i. e., the evaluation of

$$\int_a^b p(x)P_n^2(x)dx,$$

in general requires considerable computation, even in the simplest cases (e. g. Legendre, Laguerre, Hermite polynomials), *where we ordinarily use the known explicit expression of the polynomials involved.* Very often we have to use other characteristic properties (differential equations, difference equations, generating functions, etc.) if we wish to know more of the polynomials concerned, for example, if we wish to evaluate S_n, the sum of the roots of $\phi_n(x)$, or $\phi_n(a)$, or the elements of the "corresponding" and "associated" continued fractions

$$(4)\qquad \int_a^b \frac{p(y)}{x-y}dy \sim \cfrac{b_1(p)}{x - \cfrac{b_2(p)}{1 - \cfrac{b_3(p)}{x - \ddots}}}$$

$$= \cfrac{\lambda_1(p)}{x - c_1(p) - \cfrac{\lambda_2(p)}{x - c_2(p) - \ddots}},$$

of which the first exists in case $a \geqq 0$.

The object of this article is to develop an extremely simple and elementary method,* which in cases of polynomials of Legendre etc., gives directly, with practically no computation, the *exact value* of a_n, S_n, $\phi_n\binom{a}{b}$, λ_n, $\cdots$, and in more general cases the *asymptotic expressions* (for $n \to \infty$) of these quantities. *The method does not use any property of our polynomials, except the relations* (3), *and these only in the simplest manner.*

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* A method similar with regard to the underlying idea has been developed for a different purpose by W. Stekloff in his paper *Sur une application de la théorie de fermeture. . . .* , Mémoires de l'Académie des Sciences, St. Petersburg, vol. 33 (1914), pp. 1–59.

2. *Fundamental Formulas*. The following formula follows at once from (3):

$$(5)\qquad \int_a^b p(x)\phi_n(x)\sum_{i=0}^{n+1} g_i x^i dx = \frac{g_{n+1}S_{n+1}(p) + g_n}{a_n(p)},$$

where the g_i are arbitrary.

In connection with the continued fractions (4), we shall use also the following formulas:

$$(6)\qquad \lambda_{n+1} = \frac{a_{n-1}^2(p)}{a_n^2(p)},$$

$$(7)\qquad c_{n+1}(p) = S_{n+1}(p) - S_n(p) = \int_a^b p(x)x\phi_n^2(x)dx,\text{*}$$

$$(8)\qquad b_{2n}(p) = \frac{a_{n-1}^2(p)}{a_n^2(px)},\quad b_{2n+1}(p) = \frac{a_{n-1}^2(px)}{a_n^2(p)}.$$

Assume now the existence of the derivative $p'(x)$ in (a, b).† In the fundamental formula

$$F(b) - F(a) \equiv F(x)\Big]_a^b = \int_a^b F'(x)dx,$$

replace $F(x)$ by

$$p(x)\phi_n^2(x)x^\epsilon,\qquad p(x)\phi_n(x)\phi_{n+1}(x)x^\epsilon,$$

with $\epsilon = 0, 1$; and we get immediately, *using* (5) *only*,

$$(9)\qquad p(x)\phi_n^2(x)\Big]_a^b = \int_a^b p'(x)\phi_n^2(x)dx,$$

$$(10)\qquad p(x)x\phi_n^2(x)\Big]_a^b = \int_a^b p'(x)x\phi_n^2(x)dx + 2n + 1,$$

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* J. Chokhate (J. A. Shohat), *Sur le développement de l'intégrale* $\int_a^b [p(y)/(x-y)]\, dy\ldots$, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25–46; pp. 29–30.

† This assumption is made here for the sake of simplicity and can be greatly modified.

$$(11)\qquad p(x)\phi_n(x)\phi_{n+1}(x)\Big]_a^b = \int_a^b p'(x)\phi_n(x)\phi_{n+1}(x)dx + (n+1)\frac{a_{n+1}(p)}{a_n(p)},$$

$$(12)\qquad p(x)x\phi_n(x)\phi_{n+1}(x)\Big]_a^b = \int_a^b p'(x)x\phi_n(x)\phi_{n+1}(x)dx + \frac{S_{n+1}(p)a_{n+1}(p)}{a_n(p)}.$$

These formulas give directly the values of $a_n(p)$, $S_n(p)$, $\phi_n(\substack{a\\b})$, $\lambda_n(p)$, $\cdots$, if the nature of $p'(x)$ is known.

3. *Illustrations.* In order to illustrate the application of formulas (9)–(12) we consider some special important cases.

A. *Legendre polynomials*: $(a, b) = (-1, 1)$, $p(x) \equiv 1$.

$$(13)\qquad \phi_n^2(1) = \phi_n^2(-1) = \frac{2n+1}{2}, \quad \text{[from (9),(10)]};$$

$$(14)\qquad \frac{a_{n+1}}{a_n} = \left(\frac{(2n+1)(2n+3)}{(n+1)^2}\right)^{1/2}, \quad \text{[from (11)]};$$

$$(15)\qquad S_{n+1} = 0, \qquad \text{[from (12)]}.$$

Replacing in (14) n by $0, 1, 2, \cdots, n-1$, and multiplying, we find

$$(16)\qquad a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left(\frac{2n+1}{2}\right)^{1/2},$$

since we have, in general,

$$\frac{1}{a_0^2(p)} = \int_a^b p(x)dx;$$

and therefore, for the ordinary Legendre polynomial $P_n(x)$, by (13) or (16),

$$\int_{-1}^1 P_n^2(x)dx = \frac{2}{2n+1}.$$

B. *Generalized polynomials of Laguerre*: $(a, b) = (0, \infty)$; $p(x) = e^{-x}x^{\alpha-1} (\alpha > 0)$; $p'(x) = -p(x) + [(\alpha - 1)p(x)/x]$. We get similarly

$$c_{n+1} = 2n + \alpha,\ S_{n+1} = (n+1)(n+\alpha), \qquad [\text{from } (10), (7)]; \tag{17}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{[(n+1)(n+\alpha)]^{1/2}}; \tag{18}$$
$$a_n = \frac{1}{[\Gamma(n+1)\Gamma(n+\alpha)]^{1/2}}, \quad [\text{from } (12)].$$

The formula (9) gives for Laguerre polynomials $(\alpha = 1)$

$$\phi_n^2(e^{-x}; 0) = 1. \tag{19}$$

C. *Polynomials of Hermite*: $(a, b) = (-\infty, \infty)$; $p(x) = e^{-x^2}$.

$$c_{n+1} = 0, \qquad S_{n+1} = 0 \quad [\text{from } (9) \text{ or } (12)], \tag{20}$$

$$\frac{a_{n+1}}{a_n} = \left(\frac{2}{n+1}\right)^{1/2}, \tag{21}$$
$$a_n = \left(\frac{2^n}{\Gamma(n+1)}\right)^{1/2} = \left(\frac{2^n}{\pi^{1/2}\Gamma(n+1)}\right)^{1/2}, \quad (\text{from } (11)].$$

The above formulas, combined with (6)–(8), give directly

$$\int_{-1}^{1} \frac{p(y)}{x-y}dy = \cfrac{\lambda_1}{x - \cfrac{\lambda_2}{x - \ddots}}, \tag{22}$$

with

$$\lambda_{n+1} = \frac{n^2}{(2n-1)(2n+1)},$$

$$\int_0^\infty \frac{e^{-y}y^{\alpha-1}dy}{x-y} (\alpha > 0) = \cfrac{b_1}{x - \cfrac{b_2}{1 - \cfrac{b_3}{x - \ddots}}} \tag{23}$$
$$= \cfrac{\lambda_1}{x - c_1 - \cfrac{\lambda_2}{x - c_2 - \ddots}},$$

with $b_{2n}=n+\alpha-1$, $b_{2n+1}=n$, $\lambda_{n+1}=n(n+\alpha-1)$, $c_{n+1}=2n+\alpha$;

$$(24)\qquad \int_{-\infty}^{\infty}\frac{e^{-y^2}dy}{x-y}=\cfrac{\lambda_1}{x-c_1-\cfrac{\lambda_2}{x-c_2-\ddots}},$$

with $\lambda_{n+1}=n/2$.

4. *Generalizations.* In more general cases we replace in the formulas (9)–(12) above, the expression $p(x)\phi_n^2(x),\cdots$ by $p(x)f(x)\phi_n^2(x),\cdots$, with a properly chosen $f(x)$. Thus we get, in a very elementary way (even with $f(x)\equiv 1$), asymptotic expressions (for $n\rightarrow\infty$) of $\phi_n\binom{a}{b}$, a_n, λ_n, $\cdots$, which lead to new important results. This will be developed elsewhere. We shall give one illustration only.

Let us assume that (a, b) is finite, and that $p(x)=(x-q)^{\alpha-1}q(x)$ $(\alpha>0)$ with $q'(x)/q(x)$ bounded on (a, b), and $q(b)\neq 0$.* Then a very simple combination of (9), (10)† gives

$$\phi_n(b)=\left(\frac{2n+1}{(b-a)^\alpha q(b)}\right)^{1/2}\left[1+o\left(\frac{1}{n}\right)\right].$$

Furthermore,

$$\phi_n^{(k)}(b)=H_k n^{2k+1/2}(1+o(1)),\qquad (k\text{ finite, }=1,2,\cdots),$$

where $H_k(>0)$ does not depend on n.

Legendre polynomials are included here as a very special case.

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* See the second footnote on page 429.

† Take $(10)-a\times(9)$.