GENERALIZATION OF THE BELTRAMI EQUATIONS TO CURVED n-SPACE*

BY G. E. RAYNOR

Let S be a curved n-space in which the linear element is given by the equation

$$ds^2 = \sum E_{ij}dx_idx_j, \qquad (i,j = 1,2, \cdots ,n). \tag{1}$$

Without loss of generality, we may suppose

$$E_{ij} = E_{ji}. \tag{2}$$

Also let $U^{(i)}$, $(i=1, 2, \cdots, n)$, be a set of n independent functions of $x_1, x_2, \cdots, x_n$.

We shall say that the $U^{(i)}$ are isothermal in S provided they satisfy the relation

$$\sum (dU^{(i)})^2 = \lambda \sum E_{ij}dx_idx_j, \tag{3}$$

where λ is a function of the x_i only.

If in (3) we express the $dU^{(i)}$ in terms of the differentials of $x_1, x_2, \cdots, x_n$ it follows from the independence of these differentials that the coefficients of corresponding terms on the two sides of the equation are equal and we obtain the $n(n+1)/2$ equations

$$\sum_{k=1}^{n} U_{xi}^{(k)} U_{xj}^{(k)} = \lambda E_{ij}. \tag{4}$$

Let D be the discriminant of the quadratic differential form in (1) and suppose it to be written as a determinant

$$|E_{ij}|, \tag{5}$$

in which E_{ij} is the element in the ith row and jth column. If each element of (4) be multiplied by λ and if for λE_{ij} be substituted its equal given by the left side of (4), we

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readily see that the resulting determinant is the square of the Jacobian

$$J = \frac{\partial(U^{(1)}, U^{(2)}, \cdots, U^{(n)})}{\partial(x_1, x_2, \cdots, x_n)}.$$

Hence we have

$$(6) \qquad J = \lambda^{n/2} D^{1/2}.$$

In all that follows we shall suppose J to be written as a determinant in which $U^{(i)}_{x_j}$, the derivative of $U^{(i)}$ with respect to x_j, is the element of J in the ith row and jth column.

Multiply both sides of (6) by $U^{(i)}_{x_j}$, on the left letting the factor go into the ith row of J. Now if we multiply each row of J, other than the ith, by its jth element and add the corresponding products to the elements in the ith row, (6) becomes by means of (2) and (4)

$$(7) \qquad \lambda J_{ij} = \lambda^{n/2} D^{1/2} U^{(i)}_{x_j},$$

where J_{ij} is the determinant obtained from J by replacing its ith row by the jth row of D. From (6) and (7) we obtain

$$(8) \qquad U^{(i)}_{x_j} = \frac{J_{ij}}{J^{(n-2)/n} D^{1/n}}, \qquad (i,j = 1,2, \cdots, n).$$

This last set of n^2 equations are of the form obtained, by a different method, by Hedrick and Ingold* for curved 3-space. Their equation in our notation may be written

$$(9) \qquad U^{(i)}_{x_j} = P J_{ij}$$

where P is an unspecified factor of proportionality. However, it may be seen as follows that equations (9) are equivalent to (8). Replace each element of J by its expression given by (9) and we obtain

$$(10) \qquad J = P^n |J_{ij}|.$$

* Transactions of this Society, vol. 27 (1925), p. 561.

Now each element J_{ij} of the determinant $|J_{ij}|$ is a determinant which has one row of D in it. If we expand this determinant by cofactors with respect to the elements of this row we find that $|J_{ij}|$ breaks up into the product

$$DA_J$$

where A_J is the adjoint of J. Hence (10) becomes

$$J = P^n D J^{n-1},$$

and

$$P = \frac{1}{J^{(n-2)/n} D^{1/n}}.$$

If $n=2$, equations (8) become

$$U_{x_j}^{(i)} = \frac{J_{ij}}{D^{1/2}}, \qquad (i,j = 1,2), \tag{11}$$

which are precisely the well known Beltrami equations of differential geometry. These equations have the property that the derivatives of either one of the $U^{(i)}$ are expressed in terms of the E_{ij} and the derivatives of the other U only. This is not the case for $n>3$ in (8), since J on the right contains $U_{x_j}^{(i)}$ which appears on the left. To obtain a more desirable form we proceed as follows.

From the sub-set of the equations in (8) obtained by keeping i fixed, we get by taking ratios,

$$U_{x_k}^{(i)} = \frac{J_{ik}}{J_{ij}} U_{x_j}^{(i)}, \qquad (k \neq j). \tag{12}$$

After substituting these expressions for $U_{x_k}^{(i)}$ in the ith row of J on the right side of (8), $U_{x_j}^{(i)}$ can be removed as a factor from this row and solving the resulting equation for $U_{x_j}^{(i)}$ we obtain

$$U_{x_j}^{(i)} = \frac{J_{ij}}{M_i^{(n-2)/(2n-2)} D^{1/(2n-2)}}, \tag{13}$$

where by M_i we mean the determinant obtained from J by replacing its ith row by the ith row of the determinant $|J_{ij}|$.

Equations (13) contain none of the derivatives of $U^{(i)}$ on the right and hence we have all the partial derivatives of $U^{(i)}$ expressed in terms of the E_{ij} and $U^{(k)}_{x_j}$, $(k \neq i)$. Hence (13) is the desired generalization of the Beltrami equations to curved n-space.

It can readily be shown, conversely, that if a set of functions satisfy (8) or (13) they also satisfy (3).*

We now proceed to find the differential equation satisfied by each of the $U^{(i)}$ singly. Let C_{ij} denote the cofactor of the element in the ith row and jth column of J. Then in (8), if we expand the J_{ij} by cofactors with respect to the row of E's contained in them, (8) can be written

$$(14) \qquad \sum_{j=1}^{n} E_{kj} C_{ij} = J^{(n-2)/n} D^{1/n} U^{(i)}_{x_k}, \qquad (i, k = 1, 2, \cdots, n).$$

If out of the above set of n^2 equations we solve the set of n, obtained by holding i fixed, for the C_{ij} we get

$$(15) \qquad C_{ij} = \frac{N_{ij} J^{(n-2)/n}}{D^{(n-1)/n}},$$

where N_{ij} is the determinant obtained from D by substituting the ith row of J for the jth row of D. If now J in the last equation be expanded with respect to cofactors of the ith row, (15) becomes

$$(16) \qquad C_{ij} = \frac{N_{ij} \left\{ \sum_{k=1}^{n} U^{(i)}_{x_k} C_{ik} \right\}^{(n-2)/n}}{D^{(n-1)/n}}, \qquad (j = 1, 2, \cdots, n).$$

From (16) we get, by taking ratios,

$$(17) \qquad C_{ik} = \frac{N_{ik}}{N_{ij}} C_{ij}, \qquad (k \neq j).$$

Substituting these values for C_{ik}, in the right of (16), and, solving the resulting equation for C_{ij}, we have

* See Hedrick and Ingold, loc. cit., p. 562.

$$(18) \qquad C_{ij} = \frac{N_{ij}\left\{ \sum_{k=1}^{n} U_{xk}^{(i)} N_{ik} \right\}^{(n-2)/2}}{D^{(n-1)/2}}.$$

Now by a well known property of Jacobians,*

$$(19) \qquad \sum_{j=1}^{n} \frac{\partial C_{ij}}{\partial x_j} = 0.$$

Hence, if in (19) the expressions on the right of (18) be substituted for C_{ij}, we will have the differential equation satisfied by $U^{(i)}$ alone. It is readily seen that the form of this equation is independent of the index (i) and hence the n functions

$$U^{(1)}, U^{(2)}, \cdots, U^{(n)}$$

satisfy the same differential equation, which may be looked upon as a generalization of Laplace's equation to curved n-space.

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THE NON-EXISTENCE OF A CERTAIN TYPE OF REGULAR POINT SET†

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In a paper not yet published,‡ I have shown that a regular§ connected point set which consists of more than one point and remains connected upon the omission of any connected subset, is a simple closed (Jordan) curve. As a simple closed curve is a bounded point set, it is clear that there does not exist any unbounded regular connected point set which remains connected upon the omission of any connected subset.

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* Muir, *Theory of Determinants*, vol. 2, p. 230.

† Presented to the Society, December 29, 1926.

‡ See, however, this Bulletin, vol. 32 (1926), p. 591, paper No. 35.

§ That is, connected im kleinen.