ON THE GEOMETRY OF LINEAR DISPLACEMENT*

BY D. J. STRUIK

1. *Introduction.* The first to consider the displacement of a vector to a point at an infinitesimal distance in a non-euclidean geometry seems to have been L. E. J. Brouwer for the case of space of constant curvature.† Its importance for differential geometry was pointed out for the first time, however, by Levi-Civita (1917. 1),‡ for the case of a hypersurface immersed in an euclidean space, that is, the case of an n-dimensional Riemannian displacement. It was with this paper of Levi-Civita that the new development of differential geometry was started. Weyl (1918. 1 ; 1918. 3) first pointed out the way in which Levi-Civita's results could be generalized and he originated what we call *the Weyl displacement.* Even before Weyl and Levi-Civita, generalizations of analogous character had been suggested by Hessenberg (1916. 1). The geometry of linear displacement was developed in the papers of König (1919. 1 ; 1920. 1), Eddington (1921. 1 ; 1923. 4), Weyl (1918. 1 ; 1918. 3), and Schouten (1922. 1). Schouten gave a general classification of the different geometries defined by linear displacements, which he published later in his book (1924. 9). We adopt in general Schouten's notation.

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† L. E. J. Brouwer, *Het krachtveld der nieteuklidische negatief gekromde ruimten*, Verslagen Akademie Amsterdam, vol. 15 (1906), pp. 75–94 ≡ *The force field of the non-euclidean spaces with negative curvature*, Proceedings Academy Amsterdam, vol. 9 (1906), pp. 116–133. See however also a remark of H. Poincaré in A. Dall' Acqua, *Sulla teoria delle congruenze di curve in una varietà qualunque a tre dimensioni*, Annali di Matematica, (3), vol. 6 (1901), pp. 1–40.

‡ Such references refer to the bibliography at the end of this paper (see p. 558). Independently of Levi-Civita, Schouten obtained analogous results (1918.4).

That euclidean geometry is a special case of Riemannian geometry had been shown by Riemann. It could now be pointed out by König (1919. 1 ; 1920. 1) and Schouten (1923. 7 ; 1924. 9) that affine geometry, the differential geometry of which had been recently developed by Blaschke, Pick, Radon, and others, had the same relation to the affine displacement, introduced by Weyl, as euclidean geometry had to the displacement of Levi-Civita. The question now arose of approaching projective and conformal geometry by the way of linear displacement. Weyl (1921. 2) was successful in this. Cartan, who had given an introduction to the geometry of linear displacement starting from a different point of view in a series of papers (printed in 1922 and 1923), also approached projective and conformal geometries in his own way (papers of 1924). Schouten (1924. 7; 1924. 8) pointed out the relation between Cartan's and König's results.

Another way of attacking the problems of a linear displacement was found by Eisenhart and Veblen (1922. 2). These authors do not start with the displacement, but define it by a set of partial differential equations of the second order representing the paths (geodesic lines) of the displacement. It was especially the affine displacement that was the object of their investigation and of those of T. Y. Thomas and J. M. Thomas (1924–26). They developed further projective and conformal geometries, and also the non-symmetric displacement. One of their results is a theory of differential invariants of these displacements. Another author who studied this part of the theory is Weitzenböck (1920. 2 ; 1922. 13). T. Y. Thomas also gave a new way to introduce projective geometry into the geometry of paths (1925. 13 ; 1926. 11).

Applications of geometries of linear displacements to the theory of relativity were made by Weyl (1918, 1919), Eddington (1921. 1 ; 1923. 4), Einstein (1923. 1 ; 1923. 2 ; 1923. 3), and others (see bibliography).

A recent application of the theory of linear displacements to the theory of semi-simple and simple continuous groups was given by Cartan and Schouten (1925. 3).

2. *The n-Dimensional Manifold and Ricci Algebra.* The geometry of linear displacement starts with the assumption of sets of points which constitute a *domain* of an *n-dimensional manifold.* Such a domain will be denoted by X_n. This assumption implies a set of postulates which are discussed by Veblen (1925. 16, pp. 132–136).* Veblen introduces points as undefined terms and asserts that there exists a way of labelling the points in a biunique way by means of ordered sets of n numbers x^ν, $\nu=1, 2, \cdots, n$. This is equivalent to the *introduction of an n-cell* as a set of points capable of supporting a euclidean geometry. We then get a set of postulates for an arbitrary geometry of linear displacements, by first giving such a set for a euclidean geometry and then postulating that the geometry under consideration be *analytically related* to that of Euclid. That means, that if A be a point of the euclidean space, and R_A be an n-dimensional region containing A, in which the points are uniquely denoted by coordinates y^ν, and R_P an analogous region containing a point P of X_n, in which the relations of geometry (as the $\Gamma^\nu_{\lambda\mu}$) are analytic functions of the coordinates x^ν, there is an analytic relation

$$y^\nu = f(x^\mu), \qquad (\mu,\nu = 1,2, \cdots ,n), \tag{1}$$

with a unique inverse.

When the coordinates x^ν of X_n undergo an analytic transformation with a unique inverse

$$'x^\nu = \phi^\nu(x^\mu), \qquad (\mu,\nu = 1,2, \cdots ,n) \tag{2}$$

the differentials of these coordinates in general suffer a homogeneous linear transformation

* See also H. Lebesgue, *Sur les correspondances entre les points de deux espaces*, Fundamenta Mathematicae, vol. 2 (1921), pp. 256–285, where other literature (e.g., Cantor, Brouwer) is cited.

$$(3) \qquad {}'dx^\nu = \frac{\partial' x^\nu}{\partial x^\mu} dx^\mu, \qquad \Delta = \left| \frac{\partial' x^\lambda}{\partial x^\mu} \right| \neq 0,$$

and these transformations build up the linear homogeneous or affine group. Hence the infinitesimal region of X_n at a point P can be considered as an infinitesimal affine space E_n in which the differentials can be regarded as representing a set of rectilinear coordinate axes. Thus we can introduce at one point of an X_n the whole algebra of Ricci calculus, which is identical with the theory of invariants and covariants of the linear homogeneous group. Hence we can speak *at one point* of X_n of contravariant, covariant, and mixed quantities (vectors and tensors), the most simple contravariant vector being dx^ν. The three principal invariant operations between these quantities are

(a) transvection (Überschiebung), for example :

$$v_\lambda w^\lambda, \; v_{\lambda\mu\nu} w^\mu, \; R_{\omega\mu\lambda}^{\cdot\cdot\cdot\;\omega}, \text{etc.};$$

(b) alternation, for example :

$$v_{[\mu} w_{\lambda]} = \tfrac{1}{2}(v_\mu w_\lambda - v_\lambda w_\mu),$$

$$v_{[\lambda\mu\nu]} = \tfrac{1}{6}(v_{\lambda\mu\nu} - v_{\lambda\nu\mu} + v_{\mu\nu\lambda} - v_{\mu\lambda\nu} + v_{\nu\lambda\mu} - v_{\nu\mu\lambda});$$

(c) mixing, for example :

$$v_{(\mu} w_{\lambda)} = \tfrac{1}{2}(v_\mu w_\lambda + v_\lambda w_\mu),$$

$$v_{(\lambda\mu\nu)} = \tfrac{1}{6}(v_{\lambda\mu\nu} + v_{\lambda\nu\mu} + v_{\mu\nu\lambda} + v_{\mu\lambda\nu} + v_{\nu\lambda\mu} + v_{\nu\mu\lambda}).$$

3. *Linear Displacement.* To begin with, there is nothing to compare the quantities at any point $P(x^\nu)$ with the quantities at another point $Q(x_1^\nu)$. At one point P we can at least compare the lengths of vectors in the same direction (for instance we can define nv^ν as having n times the length of v^ν*), of simple bivectors $v_{[\mu} w_{\lambda]}$ in the same plane direction, etc. But even this fails at different points, and we can only compare

* We shall denote vectors and tensors by their components, and speak e.g. of the *vector* v^ν instead of the *vector* with components v^ν.

the values at P and Q of a given scalar field. In Riemannian geometry we overcome the difficulty by the introduction of a quadratic differential form

$$ds^2 = g_{\lambda\mu} dx^\lambda dx^\mu, \tag{4}$$

which enables us

(a) to define at one point the length of a contravariant vector and the angle between two of them,

(b) to compare these vectors, and hence their lengths and angles at two different points P and P' an infinitesimal distance apart.

This is done by *geodesic parallelism*, introduced by Levi-Civita (1917. 1), by means of which it is possible (see also (Schouten 1918. 4)) to define the *covariant differentials* of a vector field $v^\nu(x^1, \cdots, x^n)$, uniquely defined at P and P':

$$\left\{\begin{aligned} \delta v^\nu &= dv^\nu + \left\{\begin{matrix}\lambda\mu\\ \nu\end{matrix}\right\} v^\lambda dx^\mu, \\ \delta w_\lambda &= dw_\lambda - \left\{\begin{matrix}\lambda\mu\\ \nu\end{matrix}\right\} w_\nu dx^\mu, \end{aligned}\right. \tag{5}$$

where the $\left\{\begin{smallmatrix}\lambda\mu\\ \nu\end{smallmatrix}\right\}$ depend on $g_{\lambda\mu}$ and $\partial g_{\lambda\mu}/\partial x^\nu$ as follows:

$$\left\{\begin{matrix}\lambda\mu\\ \sigma\end{matrix}\right\} = g^{\nu\sigma}\left[\begin{matrix}\lambda\mu\\ \nu\end{matrix}\right] = \frac{1}{2} g^{\nu\sigma}\left(\frac{\partial g_{\lambda\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right); \tag{6}$$

$$g^{\nu\sigma} = (\text{minor of } g_{\nu\sigma} \text{ in determ. of } g_{\lambda\mu}) \div \text{this determ.}$$

When $\delta v^\nu = 0$ or $\delta w_\lambda = 0$, we call the respective vectors at P and P' *parallel*. The formulas (5) define the *Riemannian displacement*.

In an X_n we have no fixed tensor $g_{\lambda\mu}$, but we can still give a definition of parallelism which enables us to retain part of (b). Instead of the $\left\{\begin{smallmatrix}\lambda\mu\\ \nu\end{smallmatrix}\right\}$ we therefore introduce, for one set of x^ν, n^3 arbitrary functions $\Gamma^{\nu}_{\lambda\mu}$ and n^3 arbitrary functions $\Gamma'^{\nu}_{\lambda\mu}$ and define

$$\left\{\begin{aligned} \delta v^\nu &= dv^\nu + \Gamma^{\nu}_{\lambda\mu} v^\lambda dx^\mu, \\ \delta w_\lambda &= dw_\lambda - \Gamma^{\nu}_{\lambda\mu} w_\nu dx^\mu, \end{aligned}\right. \tag{7}$$

as *covariant differentials* of the vector fields v^ν and w_λ. It is therefore necessary that a transformation of coordinates (3) imply a transformation of the $\Gamma_{\lambda\mu}^{\nu}$ and $\Gamma_{\lambda\mu}^{\prime\nu}$ as follows:

$$
(8)\quad \left\{
\begin{aligned}
\Gamma_{\omega\pi}^{\prime\kappa} &= \frac{\partial x^\lambda}{\partial' x^\omega}\frac{\partial x^\mu}{\partial' x^\pi}\frac{\partial' x^\kappa}{\partial x^\nu}\Gamma_{\lambda\mu}^{\nu} + \frac{\partial x^\mu}{\partial' x^\pi}\frac{\partial' x^\kappa}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\frac{\partial x^\nu}{\partial' x^\omega},\\
{}'\Gamma_{\omega\pi}^{\prime\kappa} &= \frac{\partial x^\lambda}{\partial' x^\omega}\frac{\partial x^\mu}{\partial' x^\pi}\frac{\partial' x^\kappa}{\partial x^\nu}\Gamma_{\lambda\mu}^{\prime\nu} + \frac{\partial x^\mu}{\partial' x^\pi}\frac{\partial' x^\kappa}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\frac{\partial x^\nu}{\partial' x^\omega},
\end{aligned}
\right.
$$

or

$$
(9)\quad \left\{
\begin{aligned}
\Gamma_{\lambda\mu}^{\nu}\frac{\partial' x^\omega}{\partial x^\nu} &= {}'\Gamma_{\rho\sigma}^{\omega}\frac{\partial' x^\rho}{\partial x^\lambda}\frac{\partial' x^\sigma}{\partial x^\mu} + \frac{\partial^{2\prime} x^\omega}{\partial x^\lambda \partial x^\mu},\\
\Gamma_{\lambda\mu}^{\prime\nu}\frac{\partial' x^\omega}{\partial x^\nu} &= {}'\Gamma_{\rho\sigma}^{\prime\omega}\frac{\partial' x^\rho}{\partial x^\lambda}\frac{\partial' x^\sigma}{\partial x^\mu} + \frac{\partial^{2\prime} x^\omega}{\partial x^\lambda \partial x^\mu}.
\end{aligned}
\right.
$$

When the Γ are transformed in this way, δv^ν and δw_λ are transformed as vectors. The transformations of the covariant and of the contravariant vectors are independent of each other. As the formulas (7) for δv^ν and δw_λ are linear in the dx^ν, the displacement they define is called the *general linear displacement*. It passes into the Riemannian displacement if a tensor $g_{\lambda\mu}$ exists so that

$$
(10)\qquad \Gamma_{\lambda\mu}^{\nu} = \Gamma_{\lambda\mu}^{\prime\nu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\},
$$

and if this tensor $g_{\lambda\mu}$ is taken as the fundamental tensor (that is, as defining the ds^2 as in (4)). This general linear displacement (7) was first introduced by Schouten (1922. 1; see also 1924. 9). Schouten also gave a set of axioms on which this displacement is founded (1924. 9, pp. 63–64). These ideas had been anticipated by König (1919. 1; 1920. 1), whose conclusions will be considered in §11.

When $\delta v^\nu = 0$, or $\delta w_\lambda = 0$, we say that the corresponding vector is moved *parallel* to itself in the sense of the defined displacement.

From the covariant differentials we pass to the covariant derivatives by the formulas

$$(11)\qquad \delta v^\nu = dx^\mu \nabla_\mu v^\nu, \qquad \delta w_\lambda = dx^\mu \nabla_\mu w_\lambda\ ;$$

hence

$$(12)\qquad \begin{cases} \nabla_\mu v^\nu = \dfrac{\partial v^\nu}{\partial x^\mu} + \Gamma^{\nu}_{\lambda\mu} v^\lambda, \\[2ex] \nabla_\mu w_\lambda = \dfrac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^{\nu}_{\lambda\mu} w_\nu. \end{cases}$$

The covariant differentials of quantities of higher degree are obtained by assuming the formula, valid for all quantities ϕ and ψ :

$$(13)\qquad \delta(\phi\psi) = (\delta\phi)\psi + \phi\delta\psi,$$

so that, for example,

$$(14)\qquad \begin{cases} \delta g_{\lambda\mu} = dg_{\lambda\mu} - \Gamma'^{\alpha}_{\lambda\beta} g_{\alpha\mu} dx^\beta - \Gamma'^{\alpha}_{\mu\beta} g_{\lambda\alpha} dx^\beta, \\[2ex] \nabla_\mu v^{\alpha}_{\cdot\beta\gamma} = \dfrac{\partial}{\partial x^\mu} v^{\alpha}_{\cdot\beta\gamma} + \Gamma^{\alpha}_{\kappa\mu} v^{\kappa}_{\cdot\beta\gamma} - \Gamma'^{\kappa}_{\beta\mu} v^{\alpha}_{\cdot\kappa\gamma} - \Gamma'^{\kappa}_{\gamma\mu} v^{\alpha}_{\cdot\beta\kappa}. \end{cases}$$

The covariant differential of a scalar field $p(x^\nu)$ is the ordinary differential

$$(15)\qquad \delta p = dp, \qquad \nabla_\mu p = \frac{\partial p}{\partial x^\mu}\cdot$$

4. *The Fields $C^{\cdot\cdot\nu}_{\lambda\mu}$ and $S^{\cdot\cdot\nu}_{\lambda\mu}$.* The Γ and Γ' are not tensors, as is shown by (9). However

$$(16)\qquad C^{\cdot\cdot\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\lambda\mu}$$

has tensor character (Schouten 1922. 1), as is shown by (9), or by the formula

$$(17)\qquad \Delta_\mu A^{\nu}_{\lambda} = \frac{\partial}{\partial x^\mu} A^{\nu}_{\lambda} + \Gamma^{\nu}_{\alpha\mu} A^{\alpha}_{\lambda} - \Gamma'^{\alpha}_{\lambda\mu} A^{\nu}_{\alpha} = \Gamma^{\nu}_{\lambda\mu} - \Gamma'^{\nu}_{\lambda\mu},$$

where

$$(18)\qquad A^{\nu}_{\lambda} \begin{cases} = 0, & \nu = \lambda \\ = 1, & \nu = \lambda \quad \text{(not summed)} \end{cases}$$

is the unit tensor (often called the Kronecker symbol and denoted by δ_λ^ν *).

It plays a role in the covariant differentiation of a transvection, for example,

$$\delta(v_\lambda w^\lambda) = (\delta v_\lambda) w^\lambda + v_\lambda \delta w^\lambda - C_{\mu\lambda}^{\cdot\cdot\nu} v_\nu w^\lambda dx^\mu. \tag{19}$$

When we consider geometries in which the equation

$$v^\lambda w_\lambda = 0 \tag{20}$$

remains invariant under displacement, we need the following form for $C_{\lambda\mu}^{\cdot\cdot\nu}$:

$$C_{\lambda\mu}^{\cdot\cdot\nu} = C_\mu A_\lambda^\nu. \tag{21}$$

In this case the *incidence* of the point represented by v^ν, and the affine E_{n-1} represented by w_λ, remains invariant under displacement. If not only incidences are invariant, but also the transvection itself, we have $C_\mu = 0$, or

$$C_{\lambda\mu}^{\cdot\cdot\nu} = 0. \tag{22}$$

This case (22) is assumed by most authors. The independence of the displacements in the contravariant and covariant cases, characterized by $C_{\lambda\mu}^{\cdot\cdot\nu} \neq 0$, is considered only in the papers of König, Schouten, and Dienes (1924. 24). In the following pages we shall asume in general that (22) is satisfied. For this case Dienes considers the integrals of (12), which lead to functions of lines.

In Schouten's† book (1924. 9) formulas for the case $C_{\lambda\mu}^{\cdot\cdot\nu} \neq 0$ are given.

* For vectors and tensors we always use Latin letters in this paper.

† Schouten also deals with another generalization. He admits a definition of covariant vectors by means of an equation

$$'w_\lambda = \tau^{-1} \frac{\partial x^\nu}{\partial' x^\lambda} w_\nu\ ; \quad w_\lambda = \tau \frac{\partial' x^\nu}{\partial x^\lambda} w_\nu,$$

where $\tau(x^\nu)$ is an arbitrary function. For $\tau = 1$ we get the ordinary definition of a covariant vector. The meaning of this definition is trivial in an E_n, but not in an X_n. He calls it the *alteration of covariant measure*. An analogous alteration of contravariant measure is not possible unless the differentials dx^ν cease to be exact differentials.

We now observe that the expressions

$$S_{\lambda\mu}^{\cdot\cdot\nu} = \tfrac{1}{2}(\Gamma_{\lambda\mu}^{\nu} - \Gamma_{\mu\lambda}^{\nu}) \tag{23}$$

have tensor character, as is shown by (9) or by the formula

$$\delta_2 d_1 x^\nu - \delta_1 d_2 x^\nu = 2 d_1 x^\lambda d_2 x^\mu S_{\lambda\mu}^{\cdot\cdot\nu}. \tag{24}$$

Thus if d_1x^ν is moved parallel to itself (in the sense of the displacement) in the direction d_2x^ν and in the same way d_2x^ν in the direction d_1x^ν we do not get in general a closed quadrilateral. Only when $S_{\lambda\mu}^{\cdot\cdot\nu}=0$ (except for quantities of higher order) do we always obtain a closed figure. In the case of $S_{\lambda\mu}^{\cdot\cdot\nu}\neq 0$, Eddington speaks of "infinitely crinkled," and Cartan of "torsion." The case $S_{\lambda\mu}^{\cdot\cdot\nu}\neq 0$ we will call *non-symmetric*, and the case $S_{\lambda\mu}^{\cdot\cdot\nu}=0$ *symmetric*. In this last case

$$\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\mu\lambda}^{\nu}, \tag{25}$$

which expresses the *Christoffel symmetry*. An intermediate case is that in which

$$S_{\lambda\mu}^{\cdot\cdot\nu} = S_{[\lambda} A_{\mu]}^{\nu} = (S_\lambda A_\mu^\nu - S_\mu A_\lambda^\nu)/2. \tag{26}$$

Then the quadrilateral of d_1x^ν and d_2x^ν is closed when and only when the directions of d_1x^ν and d_2x^ν lie in the E_{n-1} of S_λ. This case is called *semi-symmetric*. Hessenberg (1916. 1), the first to introduce a geometry belonging to general linear displacement, obtained the non-symmetric case. He then specialized on the symmetric case (1916. 1, pp. 210–211). Afterwards Weyl also came to the symmetric case (1918. 1) in the paper in which he introduced the idea of displacement in the sense in which we use it here. Weyl spoke of an *affine* displacement.

The curl of a vector field w_λ is, in a non-symmetric displacement,

$$\nabla_{[\mu} v_{\lambda]} = \frac{1}{2}\left(\frac{\partial v_\lambda}{\partial x^\mu} - \frac{\partial v_\mu}{\partial x^\lambda}\right) - S_{\lambda\mu}^{\cdot\cdot\nu} v_\nu \tag{27}$$

and hence only depends on the alternating part of the $\Gamma_{\lambda\cdot}^{\nu}$.

To every asymmetric displacement belongs, in a univocal way, a symmetric one. For let $\Lambda_{\lambda\mu}^{\nu}$ be the symmetric part of $\Gamma_{\lambda\mu}^{\nu}$:

$$(28) \qquad \Lambda_{\lambda\mu}^{\nu} = \tfrac{1}{2}(\Gamma_{\lambda\mu}^{\nu} + \Gamma_{\mu\lambda}^{\nu}),$$

or

$$(29) \qquad \Gamma_{\lambda\mu}^{\nu} = \Lambda_{\lambda\mu}^{\nu} + S_{\lambda\mu}^{\cdot\cdot\nu},$$

then $\Lambda_{\lambda\mu}^{\nu}$ is transformed in the same way (8, 9) as $\Gamma_{\lambda\mu}^{\nu}$, and thus defines a displacement. Also by writing

$$(30) \qquad \begin{aligned} \delta v^{\nu} = dv^{\nu} + \Gamma_{\lambda\mu}^{\nu} v^{\lambda} dx^{\mu} &= (dv^{\nu} + \Gamma_{\lambda\mu}^{\nu} v^{\lambda} dx^{\mu}) \\ &+ S_{\lambda\mu}^{\cdot\cdot\nu} v^{\lambda} dx^{\mu}, \end{aligned}$$

and considering the vector character of $S_{\lambda\mu}^{\cdot\cdot\nu} v^{\lambda} dx^{\mu}$, we see the vector character of

$$(31) \qquad {}^{*}\delta v^{\nu} = dv^{\nu} + \Lambda_{\lambda\mu}^{\nu} v^{\lambda} dx^{\mu}.$$

This remark was made by J. M. Thomas (1926.2).

5. *Introduction of a Metric.* Again taking $C_{\lambda\mu}^{\cdot\cdot\nu} = 0$, we may observe that, in a general linear displacement, the differential of a tensor $g^{\lambda\mu}$ gives an affinor $Q_{\mu}^{\cdot\lambda\mu}$ of third degree :

$$(32) \qquad \begin{cases} \delta g^{\lambda\nu} = Q_{\mu}^{\cdot\lambda\nu} dx^{\mu}, \\ \nabla_{\mu} g^{\lambda\nu} = Q_{\mu}^{\cdot\lambda\nu}. \end{cases}$$

Then, differentiating the covariant tensor $g_{\lambda\mu}$ belonging to $g^{\lambda\mu}$, if we assume that its rank is n, we get by (6),

$$(33) \qquad \begin{cases} \delta g_{\lambda\mu\nu} = - g_{\lambda\alpha} g_{\nu\beta} Q_{\mu}^{\cdot\alpha\beta} dx^{\mu}, \\ \nabla_{\mu} g_{\lambda\nu} = - g_{\lambda\alpha} g_{\nu\beta} Q_{\mu}^{\cdot\alpha\beta}. \end{cases}$$

Suppose now, that the $\Gamma_{\lambda\mu}^{\nu}$ can be so chosen that there exists at least one tensor (of rank n) $g_{\lambda\mu}$ so that

$$(34) \qquad Q_{\mu}^{\cdot\lambda\nu} = Q_{\mu} g^{\lambda\nu},$$

hence

$$(35) \qquad \begin{cases} \delta g^{\lambda\nu} = Q_{\mu} g^{\lambda\nu} dx^{\mu}, \\ \nabla_{\mu} g^{\lambda\nu} = Q_{\mu} g^{\lambda\nu}. \end{cases}$$

In that case also

$$\delta g_{\lambda\nu} = - Q_\mu g_{\lambda\nu} dx^\mu \tag{36}$$

and we speak of a *conformal* displacement (Schouten 1924.9, p. 72). That this geometry is possible is seen from the general theorem which we will also state for the case of $C_{\lambda\mu}^{\cdot\cdot\nu} \neq 0$, *that it is possible to define in a unique way every set of* $\Gamma_{\lambda\mu}^\nu$ *and* $\Gamma_{\lambda\mu}^{\prime\nu}$, *and hence every linear displacement, by two arbitrary affinor fields* $C_{\lambda\mu}^{\cdot\cdot\nu}$ *and* $S_{\lambda\mu}^{\cdot\cdot\nu}$ *and a field* $Q_\mu^{\cdot\lambda\nu}$ *defined as the derivative of a tensor field* $g^{\lambda\nu}$ *of rank n.*

The proof can be given by the explicit elimination of $\Gamma_{\lambda\mu}^\nu$ and $\Gamma_{\lambda\mu}^{\prime\nu}$ from the equations expressing $C_{\lambda\mu}^{\cdot\cdot\nu}$, $S_{\lambda\mu}^{\cdot\cdot\nu}$ and $Q_\mu^{\cdot\lambda\nu}$ (and $g^{\lambda\nu}$) in terms of the $\Gamma_{\lambda\mu}^\nu$ and $\Gamma_{\lambda\mu}^{\prime\nu}$. The explicit formula for the case $C_{\lambda\mu}^{\cdot\cdot\nu} = 0$ is

$$\Gamma_{\lambda\mu}^\nu = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} + T_{\lambda\mu}^{\cdot\cdot\nu}, \tag{37}$$

where the Christoffel symbol must be taken as belonging to $g^{\lambda\nu}$ and

$$\left\{ \begin{aligned} T_{\lambda\mu}^{\cdot\cdot\nu} = {} & \tfrac{1}{2}(g_{\lambda\alpha} Q_\mu^{\cdot\alpha\nu} + g_{\mu\alpha} Q_\lambda^{\cdot\nu\alpha} - g^{\nu\gamma} g_{\mu\alpha} g_{\lambda\beta} Q_\gamma^{\cdot\alpha\beta}) \\ & + S_{\lambda\mu}^{\cdot\cdot\nu} - g^{\nu\beta}(g_{\lambda\alpha} S_{\beta\mu}^{\cdot\cdot\alpha} + g_{\mu\alpha} S_{\beta\lambda}^{\cdot\cdot\alpha}). \end{aligned} \right. \tag{38}$$

This theorem was first given by Schouten (1922. 1; see also 1924. 9, p. 72–74).

If now we take $Q_\mu^{\cdot\lambda\nu} = Q_\mu g^{\lambda\nu}$ in (37) and (38), we can construct a set of $\Gamma_{\lambda\mu}^\nu$ which define a conformal displacement.

In a conformal geometry every tensor

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu}, \tag{39}$$

where σ is an arbitrary function of x^ν, has a differential of the form (35) or (36). For we have

$$\left\{ \begin{aligned} \delta' g_{\lambda\nu} &= - dx^\mu \left(\frac{\partial\sigma}{\partial x^\mu} - \sigma Q_\mu \right) g_{\lambda\nu} \\ &= - \left(Q_\mu - \frac{\partial \log \sigma}{\partial x^\mu} \right) g_{\lambda\nu} dx^\mu \\ &= - Q_\mu' g_{\lambda\nu} dx^\mu, \end{aligned} \right. \tag{40}$$

where

$$Q'_\mu = Q_\mu - \frac{\partial \log \sigma}{\partial x^\mu}. \tag{41}$$

We take one of these tensors $g_{\lambda\mu}$ and define by means of it the angle between two directions v^ν, w^ν:

$$\cos\theta = \frac{g_{\lambda\mu}v^\lambda w^\mu}{\sqrt{g_{\lambda\mu}v^\lambda v^\mu}\sqrt{g_{\lambda\mu}w^\lambda w^\mu}}, \tag{42}$$

which is invariant under (39). We then call the tensors (39) the *fundamental tensors* of the displacement. We have proved by (40), that when *one tensor* $g_{\lambda\mu}$ *has a differential of the form* (36), *every tensor* $\sigma g_{\lambda\mu}$ *has a differential of this form.* It can also be shown *that only the fundamental tensors* $\sigma g_{\lambda\mu}$ *possess this property* (Schouten 1924.9, p. 220).

If in particular Q_μ is a gradient vector, there is, by (41), one fundamental tensor $g_{\lambda\mu}$ uniquely determined so that

$$\delta g_{\lambda\mu} = 0. \tag{43}$$

In this case the displacement is called *metrical.* We may remark *that one and only one linear displacement is determined by a field* $S_{\lambda\mu}^{\cdot\cdot\nu}$ *alternating in* λ *and* μ *and a tensor field* $g_{\lambda\mu}$ *of rank* n, *when* $\Gamma_{\lambda\mu}^\nu = S_{\lambda\mu}^{\cdot\cdot\nu}$ *and* $\nabla_\omega g_{\lambda\mu} = 0$. This is a special case of the preceding theorem.

When this tensor $g_{\lambda\mu}$ is taken as fundamental tensor, we can define length and angle by means of (42) and

$$ds^2 = g_{\lambda\mu}dx^\lambda dx^\mu. \tag{44}$$

Length and angle of vectors remain invariant under a metrical displacement, and only under such a displacement, as is easily shown. When in particular $\Gamma_{\lambda\mu}^\nu = \Gamma_{\mu\lambda}^\nu$, the conformal displacement passes into the *displacement of Weyl*, introduced by him (1918. 1, 3; see also 1918. 2), and the metrical displacement into the *displacement of Levi-Civita*, (1917. 1), which is the foundation of Riemannian geometry. Indeed, in this case (37) passes into

$$\Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} . \tag{45}$$

A simple example of a metrical displacement which is not symmetric was given by Cartan (1924.5) and Hessenberg (1925. 24). Take a sphere and a set of meridians on it. Define the displacement by the assumption that a contravariant vector is moved parallel to itself when the angle with the meridians is always the same. Length and angle remain invariant, hence the displacement is metrical. But it is not the displacement of Levi-Civita, hence there must be an affinor $S_{\lambda\mu}^{\cdot\cdot\nu} \neq 0$. It can easily be seen, by (66), that this metric on the sphere is even' holonomous (see §8). The geodesic lines of this displacement are the loxodromics of the sphere. For other examples see Cartan (1926. 15). For the relation to the parallelism of Clifford see Cartan (1924. 5).

Necessary conditions that a symmetric system of $\Gamma_{\lambda\mu}^{\nu}$ may be of the form (45) or of the form of Weyl geometry were given by Eisenhart-Veblen (1922. 2). Extension of their methods can be found in Schouten (1925. 18) and Veblen-Thomas (1923. 14).

6. *Geometry of Paths*. Two other ways have been pointed out by which to approach the linear displacements (with $C_{\lambda\mu}^{\cdot\cdot\nu} = 0$), the geometry of paths developed by Veblen and Eisenhart (1922. 2), and the linear displacements of points instead of vectors developed by Cartan (1923 and later, see for example, 1925. 1). The geometry of paths does not start with the displacement itself, but with the geodesic lines. A geodesic line arises when a contravariant vector or a covariant $(n-1)$-vector moves parallel to itself in its own direction. Hence the equations of a *contravariant geodesic line* are

$$v^{\mu}\nabla_{\mu}v^{\nu} = \alpha v^{\nu}, \tag{46}$$

where v^{ν} means a contravariant vector tangent to the curve $x^{\nu} = x^{\nu}(t)$, and α is a function of the x^{ν}. The equations of a *covariant geodesic line* are

$$v^{\mu}\nabla_{\mu}v_{\lambda_1 \cdots \lambda_{n-1}} = \beta v_{\lambda_1 \cdots \lambda_{n-1}}, \tag{47}$$

where $v_{\lambda_1 \ldots \lambda_{n-1}}$ is a covariant $(n-1)$-vector tangent to the curve, and β is a function of the x^ν. A covariant geodesic line is, however, also a contravariant one, whenever $C_{(\lambda\mu)}^{\cdot\cdot\nu} = C_{(\lambda} A_{\mu)}^{\nu}$ (Friedmann-Schouten 1924. 11). Since we suppose $C_{\lambda\mu}^{\cdot\cdot\nu}=0$, we have only to deal with the equation (46), or

$$\frac{d^2x^\nu}{dt^2} + \Gamma_{\lambda\mu}^{\nu}\frac{dx^\lambda}{dt}\frac{dx^\mu}{dt} = \alpha\frac{dx^\nu}{dt}\,. \tag{48}$$

From (48) we see that the equations of geodesics only depend on the symmetric part $\Lambda_{\lambda\mu}^{\nu}$ of $\Gamma_{\lambda\mu}^{\nu}$ (see end §4). The coefficient depends on the choice of the parameter t. We can make $\alpha=0$, if we introduce a new parameter l defined, by the equation

$$l = C_1 + C_2 e^{\int \alpha\, dt}\, dt, \tag{49}$$

with two constants of integration C_1 and C_2. We thus get the equations of the geodesics :

$$\frac{d^2x^\nu}{dl^2} + \Lambda_{\lambda\mu}^{\nu}\frac{dx^\lambda}{dl}\frac{dx^\mu}{dl} = 0. \tag{50}$$

Hence *we have found on a geodesic line a natural scale of measurement determined but for the choice of an origin and a multiplicative factor* (Eisenhart 1922. 5).

The equation (50) is the starting point for the geometry of paths. The integral curves are the natural generalization of the straight lines of euclidean geometry and of the geodesics of Riemannian geometry. To start with this equation, means to consider the geometry of linear displacements as a geometric theory of a particular kind of differential equations of the more general class

$$\frac{d^2x^\nu}{ds^2} = \phi^\nu\left(x, \frac{dx}{ds}\right), \tag{51}$$

where ϕ^ν is an arbitrary function of the x^ν and its derivatives. (For $n=2$, see Cartan 1924. 4, p. 232.) From this point of view the study of equations (50) is a continuation of studies started by Pascal and others.*

7. *Normal Coordinates.* It is now possible to define a special set of coordinates at each point of X_n by means of the paths defined by (50). Take a point $P(x_0^\nu)$ and a direction $(dx^\nu/ds)_0 = v_0^\nu$ through it.

Differentiations of (50) with respect to the natural parameter gives

$$\frac{d^3x^\nu}{ds^3} + \Lambda^{\nu}_{\lambda_1\lambda_2\lambda_3}\frac{dx^{\lambda_1}}{ds}\frac{dx^{\lambda_2}}{ds}\frac{dx^{\lambda_3}}{ds} = 0, \tag{52}$$

$$\frac{d^4x^\nu}{ds^4} + \Lambda^{\nu}_{\lambda_1\lambda_2\lambda_3\lambda_4}\frac{dx^{\lambda_1}}{ds}\frac{dx^{\lambda_2}}{ds}\frac{dx^{\lambda_3}}{ds}\frac{dx^{\lambda_4}}{ds} = 0, \tag{53}$$

where

$$\Lambda^{\nu}_{\lambda_1\cdots\lambda_p\omega} = \frac{\partial}{\partial x}{}_{(\omega}\Lambda_{\lambda_1\cdots\lambda_p)} - p\Lambda^{\nu}_{\alpha(\lambda_1\cdots\lambda_{p-1}}\Lambda^{\alpha}_{\lambda_p\omega)}. \tag{54}$$

Then we can obtain the expansion of the coordinates of a geodesic line;

$$\begin{aligned} x^\nu = x_0^\nu + v_0^\nu t &- \frac{1}{2!}(\Lambda^{\nu}_{\lambda\mu})_0 v_0^\lambda v_0^\mu t^2 \\ &- \frac{1}{3!}(\Lambda^{\nu}_{\lambda\mu\nu})_0 v_0^\lambda v_0^\mu v_0^\nu t^3 - \cdots, \end{aligned} \tag{55}$$

which converges for all values of t if we assume the $\Lambda^{\nu}_{\lambda\mu}$ to possess derivatives of all orders at x_0^ν, $|v_0^\nu|$ to be finite, and

$$|\Lambda^{\nu}_{\lambda_1\cdots\lambda_p}| < G, \tag{56}$$

for all values of p, where G is a given constant.

If we write

$$v_0^\nu t = y^\nu, \tag{57}$$

* E. Pascal, *Grundlagen für eine Theorie der Systeme totaler Differentialgleichungen zweiter Ordnung*, Mathematische Annalen, vol. 54 (1901), pp. 400–416; see also Comptes Rendus, vol. 130 (1900), pp. 645–647.

(56) passes into

$$x - \overset{\nu}{x_0} = y^\nu - \frac{1}{2!}(\Lambda_{\lambda\mu}^{\nu})_0 y^\lambda y^\mu - \frac{1}{3!}(\Lambda_{\lambda\mu\nu}^{\nu})_0 y^\lambda y^\mu y^\nu - \cdots . \tag{58}$$

Since the functional determinant $|\partial x^\nu/\partial y^\lambda| \neq 0$, we can solve these equations with respect to y^ν, and we get the convergent expansion

$$\left\{ \begin{aligned} y^\nu = (x^\nu - x_0^\nu) &+ \frac{1}{2!}(\Omega_{\lambda\mu}^{\nu})_0 (x^\lambda - x_0{}^\lambda)(x^\mu - \overset{\mu}{x_0}) \\ &+ \frac{1}{3!}(\Omega_{\lambda\mu\nu}^{\nu})_0 (x^\lambda - x_0{}^\lambda)(x^\mu - \overset{\mu}{x})(x^\nu - \overset{\nu}{x_0}) + \cdots , \end{aligned} \right. \tag{59}$$

where

$$\left\{ \begin{aligned} \Omega_{\lambda\mu}^{\nu} &= \Lambda_{\lambda\mu}^{\nu}, \\ \Omega_{\lambda\mu\omega}^{\nu} &= \Lambda_{\lambda\mu\omega}^{\nu} + 3\Omega_{\alpha(\lambda}^{\nu}\Lambda_{\mu\omega)}^{\alpha}, \\ \Omega_{\lambda_1\lambda_2\lambda_3\lambda_4}^{\nu} &= \Lambda_{\lambda_1\lambda_2\lambda_3\lambda_4}^{\nu} + 4\{\Omega_{\alpha(\lambda_1}^{\nu}\Lambda_{\lambda_2\lambda_3\lambda_4)}^{\alpha} \\ &\qquad + \Omega_{\alpha\beta}^{\nu}\Lambda_{(\lambda_1\lambda_2}^{\beta}\Lambda_{\lambda_3\lambda_4)}^{\alpha} + \Omega_{\alpha(\lambda_1\lambda_2}^{\nu}\Lambda_{\lambda_3\lambda_4)}^{\alpha}\} . \end{aligned} \right. \tag{60}$$

Through (58) and (59) new coordinates are defined in a finite region around $P(x_0^\nu)$, the *normal coordinates*. They are the natural generalization of Riemann's normal coordinates for Riemannian geometry,* and were first introduced by Veblen (1922. 3). Under an arbitrary transformation (4) of the x the y^ν are transformed linearly and homogeneously, as follows from (57), and precisely as the components of a contravariant vector. In an E_n the (y^ν) would represent the radius vector of a point with respect to P, and then the *normal coordinates give a representation of the X_n on the E_n at P.*

The ordinary derivatives of a quantity at P with respect to the normal coordinates at P are components of tensors.

---

* See also G. D. Birkhoff, *Relativity and Modern Physics*, Cambridge, Harvard University Press, 1923. Birkhoff calls these coordinates *Riemannian* coordinates (p. 118). He uses the term *Normal coordinates* (p. 124) for another purpose.

For example, when $v_{\lambda\mu\nu}$ is an affinor, and $\bar{v}_{\lambda\mu\nu}$ are its components in the system of normal coordinates, then the equations

$$(61)\qquad \begin{cases} w_{\lambda\mu\nu\omega} = \left(\dfrac{\partial \bar{v}_{\lambda\mu\nu}}{\partial y^{\omega}}\right)_0, \\ w_{\lambda\mu\nu\omega_1\omega_2} = \left(\dfrac{\partial^2 \bar{v}_{\lambda\mu\nu}}{\partial y^{\omega_1}\partial y^{\omega_2}}\right)_0 \end{cases}$$

define a set of functions of x^ν which satisfy the equations

$$(62)\qquad \begin{cases} w_{\lambda\mu\nu\omega} = \nabla_\omega v_{\lambda\mu\nu}, \\ w_{\lambda\mu\nu\omega_1\omega_2} = \nabla_{\omega_1}\nabla_{\omega_2} v_{\lambda\mu\nu}. \end{cases}$$

The first derivatives with respect to the normal coordinates are identical with the symmetric part of the covariant derivative. A more complete set of formulas (62) is given by Veblen-Thomas (1923. 14, pp. 573–576).

As is easily found from (50) and (58) or from (59), $\Lambda_{\lambda\mu}^{\nu}$ vanishes for normal coordinates at P. This takes place for every transformation which has two terms in common with the right side of (59), in particular for

$$(63)\qquad x^\nu - x_0^\nu = y^\nu - \frac{1}{2!}(\Lambda_{\lambda\mu}^{\nu})_0 y^\lambda y^\mu.$$

Such a system of coordinates is called *geodesic* (Weyl 1918. 2, p. 101 of 4th ed.) or *path coordinates* (Eisenhart, 1923. 11). The expansion for $\Lambda_{\lambda\mu}^{\nu}$ in normal coordinates is

$$(64)\qquad \Lambda_{\lambda\mu}^{\nu} = M_{\lambda\mu\alpha}^{\nu} y^\alpha + \frac{1}{2!} M_{\lambda\mu\alpha\beta}^{\nu} y^\alpha y^\beta + \cdots .$$

where

$$(65)\qquad M_{\lambda\mu\alpha_1\alpha_2\cdots\alpha_p}^{\nu} = \left(\frac{\partial^p \Lambda_{\lambda\mu}^{\nu}}{\partial y^{\alpha_1}\partial y^{\alpha_2}\cdots\partial y^{\alpha_p}}\right)_0 .$$

The $\Lambda_{\lambda\mu}^{\nu}$ have no tensor character.

8. *Curvature.* If P and Q are points of a curve and v^ν is a field of vectors, the integral $\int_P^Q \delta v^\nu$ taken along s is the difference

of v^ν at Q and the vector obtained by parallel moving of the vector at P along s to Q. This integral depends not only on P and Q but also on the curve s. When it does not depend on the curve, but only on P and Q for every pair of points P and Q, the X_n is called *holonomous* (Cartan 1925. 2, see 1923. 21). If s is closed and $Q=P$, we find in general a difference after the return of the vector at P to P. This difference can be calculated easily if the closed curve s is infinitely small. We find for the difference Dv^ν at P

$$(66) \qquad Dv^\nu = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} v^\lambda f^{\mu\omega} d\sigma,$$

where $f^{\omega\mu}$ is a unit bivector in the plane element which contains s (but for differential elements of higher than the first order), $d\sigma$ is the area of the element (both in such a measure that $f^{\omega\mu}\, d\sigma$ represents the bivector of the surface element determined by s), and

$$(67) \quad R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = \frac{\partial}{\partial x^\mu}\Gamma_{\omega\lambda}^{\nu} - \frac{\partial}{\partial x^\omega}\Gamma_{\mu\lambda}^{\nu} + \Gamma_{\kappa\mu}^{\nu}\Gamma_{\lambda\mu}^{\kappa} - \Gamma_{\kappa\omega}^{\mu}\Gamma_{\lambda\mu}^{\kappa}.$$

In the same way, we find

$$(68) \qquad Dw_\lambda = - R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\, w_\nu f^{\omega\mu} d\sigma.$$

The symbol $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ is an affinor, the so called *curvature affinor* of the X_n. It always appears in investigations where the alternating part of the second derivative is considered, with $C_{\lambda\mu}^{\cdot\cdot\nu}=0$, according to the formulas

$$(69) \qquad \begin{cases} 2\nabla_{[\omega}\nabla_{\mu]}v^\nu = - R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\, v^\lambda + 2S_{\omega\mu}^{\cdot\cdot\alpha}\Delta_\alpha v^\nu, \\ \\ 2\nabla_{[\omega}\nabla_{\mu]}w_\lambda = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\, w_\nu + 2S_{\omega\mu}^{\cdot\cdot\alpha}\nabla_\alpha w_\lambda. \end{cases}$$

The interpretation (66), which for the case of a Riemannian geometry was given by Levi-Civita (1917. 1), was given for Weyl's geometry by Weyl (1918. 2). Compare for discussion and exact demonstration Synge (1923.12) and T. Y. Thomas (1926. 3). See also Eddington (1923. 4, p. 68, and p. 214 of second edition).

The effect of the operator $\nabla_{[\omega}\nabla_{\mu]}$ on quantities of higher degree is seen (see (14)) from the example

$$
(70) \qquad \begin{aligned} 2\nabla_{[\omega}\nabla_{\mu]}v_{\alpha}^{\cdot\,\beta\gamma} &= R_{\omega\mu\alpha}^{\cdots\,\rho}\, v_{\rho}^{\cdot\,\beta\gamma} - R_{\omega\mu\rho}^{\cdots\,\beta}\, v_{\alpha}^{\cdot\,\rho\gamma} \\ &\quad - R_{\omega\mu\rho}^{\cdots\,\gamma}\, v_{\alpha}^{\cdot\,\beta\rho} + S_{\omega\mu}^{\cdot\cdot\,\delta}\, \nabla_{\delta} v_{\alpha}^{\cdot\,\beta\gamma}. \end{aligned}
$$

The affinor $R_{\omega\mu\lambda}^{\cdots\,\nu}$ is subject to the following identities. From the definition equation (66) follows the *first identity*:

$$
(71) \qquad R_{\omega\mu\lambda}^{\cdots\,\nu} = -R_{\mu\omega\lambda}^{\cdots\,\nu}.
$$

Again, by calculating $\nabla_{[\omega}\nabla_{\mu}\nabla_{\lambda]}$, first by applying ∇_{ω} to $\nabla_{\mu}v_{\lambda}$ and then alternating, secondly by applying $\nabla_{[\omega}\nabla_{\mu]}$ to v_{λ} and then alternating, we find the *second identity*:

$$
(72) \qquad R_{[\omega\mu\lambda]}^{\cdots\,\nu} = 4S_{[\omega\mu}^{\cdot\cdot\,\alpha} S_{\lambda]\alpha}^{\cdot\cdot\,\nu} + 2\nabla_{[\omega}S_{\mu\lambda]}^{\cdot\cdot\,\nu}.
$$

For a symmetric displacement this passes into the form

$$
(73) \qquad R_{[\omega\mu\lambda]}^{\cdots\,\nu} = 0,
$$

or

$$
(74) \qquad R_{\omega\mu\lambda}^{\cdots\,\nu} + R_{\mu\lambda\omega}^{\cdots\,\nu} + R_{\lambda\omega\mu}^{\cdots\,\nu} = 0.
$$

Moreover, by calculating $\nabla_{[\omega}\nabla_{\mu]}\, g_{\lambda\nu}$, first by applying ∇_{ω} to $\nabla_{\mu}\, g_{\lambda\nu}$, given as in (33), and then alternating, secondly by applying $\nabla_{[\omega}\nabla_{\mu]}$ to $g_{\lambda\nu}$ as in (70), we get the *third identity*:

$$
(75) \qquad R_{\omega\mu\lambda}^{\cdots\,\alpha}\, g_{\alpha\nu} + R_{\omega\mu\nu}^{\cdots\,\alpha}\, g_{\alpha\lambda} = 2\nabla_{[\omega}Q_{\mu]\lambda\nu},
$$

where

$$
Q_{\mu\lambda\nu} = -g_{\lambda\alpha}g_{\nu\beta}Q_{\mu}^{\cdot\,\alpha\beta}.
$$

For a Riemannian geometry, this passes into

$$
(76) \qquad R_{\omega\mu\lambda\nu} = -R_{\mu\omega\lambda\nu}.
$$

In a Riemannian geometry, we can deduce from the three identities the fourth:

$$
(77) \qquad R_{\omega\mu\lambda\nu} = R_{\lambda\nu\omega\mu}.
$$

The four identities of $R_{\omega\mu\lambda}^{\cdots\nu}$ for a Riemannian geometry have been given by Riemann (in his Paris memoir, 1861); for more general cases they appear in Weyl, Eddington, Veblen, Eisenhart and Schouten's papers, in Schouten's book also for the case of $C_{\lambda\mu}^{\cdot\cdot\nu}\neq 0$ (1924. 9, p. 87). The dependence of the fourth identity on the other three was first pointed out by Ricci* (see also Hessenberg 1916. 1). It also follows as a corollary from a general result of T. Y. Thomas (1926. 10).

For the derivative of $R_{\omega\mu\lambda}^{\cdots\nu}$ also an identity exists, which can be obtained, for example, by calculating $\nabla_{[\xi}\nabla_{\omega}\nabla_{\mu]}v_{\lambda}$ first by applying ∇_{ξ} on $\nabla_{[\omega}\nabla_{\mu]}v_{\lambda}$, secondly by applying $\nabla_{[\xi}\nabla_{\omega]}$ on $\nabla_{\mu}q_{\lambda}$ and then alternating on ξ, ω, μ. For $C_{\lambda\mu}^{\cdot\cdot\nu}=0$ it is

$$\nabla_{[\xi}R_{\omega\mu]\lambda}^{\cdots\nu} = -\,2S_{[\omega\mu}^{\cdot\cdot\alpha}R_{\xi]\alpha\lambda}^{\cdots\nu}. \tag{78}$$

This identity is called *Bianchi's identity*. For a symmetric displacement it becomes

$$\nabla_{[\xi}R_{\omega\mu]\lambda}^{\cdots\nu} = 0. \tag{79}$$

Bianchi published this identity in 1902 for a Riemannian geometry,† though it can be found in earlier papers of Padova (1889), who refers to a letter of Ricci.‡ Bach (1921. 3) proved it for a geometry of Weyl, Veblen (1922. 3) and Schouten (1923. 6) for symmetric displacements; Weitzenböck (1923. 25, p. 357) deduced (78). The general formulas for $C_{\lambda\mu}^{\cdot\cdot\nu}\neq 0$ can be found in Schouten (1924. 9, p. 91). (See also Schouten-Struik 1924. 10.) Cartan obtained (78) in another way (1923. 21, vol. 40, p. 373).

From $R_{\omega\lambda\mu}^{\cdots\nu}$, the following affinors can be deduced:

$$\begin{cases} R_{\mu\lambda} = R_{\omega\mu\lambda}^{\cdots\omega} \\ V_{\omega\mu} = R_{\omega\mu\lambda}^{\cdots\lambda} \\ F_{\mu\lambda} = R_{[\mu\lambda]} = R_{\omega[\mu\lambda]}^{\cdots\omega}. \end{cases} \tag{80}$$

* G. Ricci, *Sulla determinazione di varietà, dotate di proprietà intrinseche date a priori*, Rendiconti dei Lincei, (5), vol. 19^II (1910), pp. 85–90.

† L. Bianchi, *Sui simboli a quattro indice e sulla curvatura di Riemann*, Rendiconti dei Lincei, (5), vol. 11^II (1902), pp. 3–7.

‡ E. Padova, *Sulle deformazioni infinitesime*, Rendiconti dei Lincei, (4), vol. 5^I (1889), pp. 174–178.

In a Riemannian geometry $V_{\omega\mu}$ and $F_{\mu\lambda}$ vanish, and $R_{\mu\lambda}$ is symmetric. We then can build up a scalar

$$R = g^{\lambda\mu}R_{\mu\lambda} = g^{\lambda\mu}g^{\omega\nu}R_{\omega\mu\lambda\nu}. \tag{81}$$

From the differential equations for the quantities (80), obtained by contraction from the identity of Bianchi, we mention only

$$\nabla_\mu G^{\mu\lambda} = 0, \qquad G_{\mu\lambda} = R_{\mu\lambda} - \tfrac{1}{2}Rg_{\mu\lambda}, \tag{82}$$

which only holds for a Riemannian geometry. It was obtained by Ricci* and is one of the principal formulas of Einstein's theory of relativity. The equation

$$R_{\omega\mu\lambda}^{\cdots\lambda} = V_{\omega\mu} = 0 \tag{83}$$

has a simple geometric meaning. It means that the displacement conserves the volume of an n-dimensional element after a circuit along a closed curve. Hence we can introduce the conception of *volume* in the X_n, and there exists an integral invariant

$$\int U_{\lambda_1\cdots\lambda_n} f^{\lambda_1\cdots\lambda_n} d\tau_n, \tag{84}$$

where $U_{\lambda_1\cdots\lambda_n}$ is an arbitrary n-vector, and $f^{\lambda_1\cdots\lambda_n}$ a unit n-vector, $d\tau_n = dx^1 \cdots dx^n$ (Eddington 1921. 1, page 111; Eisenhart 1922.5; and Veblen 1923. 8). In normal coordinates, hence for a symmetric displacement, the curvature affinor arises from the expansion of $\Lambda_{\lambda\mu}^{\nu}$ in normal coordinates (64) in the form

$$(R_{\omega\mu\lambda}^{\cdots\nu})_0 = \left(\frac{\partial}{\partial y^\mu}\Lambda_{\lambda\omega}^{\nu}\right)_0 - \left(\frac{\partial}{\partial y^\omega}\Lambda_{\lambda\mu}^{\nu}\right)_0. \tag{85}$$

From this property the two first identities and the identity of Bianchi are easily proved (Veblen-Thomas 1923. 14, pp.

* G. Ricci, *Sulle superficie geodetiche in una varietà qualunque e in particolare nella varietà a tre dimensioni*, Rendiconti dei Lincei, (5), vol. 12 (1903), pp. 409–420.

579–580). The holonomous case is characterized by $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}=0$. An example is the euclidean case.

9. *Projective Transformations of the Paths.* We consider a symmetric displacement with $C_{\lambda\mu}^{\cdot\cdot\nu}=0$ (affine displacement). We then can ask if there are transformations of the $\Gamma_{\lambda\mu}^{\nu}$, which leave the paths invariant. They are found to be

$$(86)\qquad {}'\Gamma_{\lambda\mu}^{\nu} = \Gamma_{\lambda\mu}^{\nu} + A_{\lambda}^{\nu}p_{\mu} + A_{\mu}^{\nu}p_{\lambda},$$

where p_λ is an arbitrary covariant vector. (Weyl 1921. 2; Veblen-Thomas 1926. 4, p. 281.) Such transformations are called *projective*. Under such a transformation the curvature affinor $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ passes into

$$(87)\qquad \begin{aligned} {}'R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} &- 2A_{\lambda}^{\nu}\nabla_{[\omega}p_{\mu]} \\ &+ 2A_{[\omega}\nabla_{\mu]}p_{\lambda} + 2A_{[\omega}^{\nu}p_{\mu]}p_{\lambda}. \end{aligned}$$

From (87) follows

$$(88)\qquad {}'R_{\omega\mu\lambda}^{\cdot\cdot\cdot\lambda} = {}'V_{\omega\mu} = V_{\omega\mu} - 2(n+1)\nabla_{[\omega}p_{\mu]}.$$

This equation is completely integrable on account of Bianchi's identity; hence we find

$$(89)\qquad \nabla_{[\xi}R_{\omega\mu]\lambda}^{\cdot\cdot\cdot}v^{\lambda} = \Delta_{[\xi}v_{\omega\mu]} = 0,$$

so that we see, that *an affine displacement can always be projectively transformed into a displacement which preserves volume* (Eisenhart, 1922. 5; Veblen, 1922. 6; 1922. 13).

The projective theory of affine displacements is the theory of those tensors which are unchanged by the transformation (86). See, however, a remark of Veblen-Thomas (1926. 4). One of these tensors, the *projective curvature tensor*, was found by Weyl (1921. 2); it is

$$(90)\qquad P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - 2P_{[\omega\mu]}A_{\lambda}^{\nu} + 2A_{[\omega}^{\nu}P_{\mu]\lambda},$$

where

$$(91)\qquad P_{\mu\lambda} = -\frac{1}{n^2-1}(nR_{\mu\lambda} + R_{\lambda\mu}).$$

For $n=1$ and $n=2$, $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ vanishes identically. It satisfies, for $n>2$, the identities

$$
(92) \qquad \begin{cases} P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = -P_{\mu\omega\lambda}^{\cdot\cdot\cdot\nu}, \\ P_{[\omega\mu\lambda]}^{\cdot\cdot\cdot\nu} = 0, \\ \nabla_{[\xi}P_{\omega\mu]\lambda}^{\cdot\cdot\cdot\nu} = \dfrac{1}{n-2}\nabla_{\alpha}A^{\nu}{}_{[\omega}P_{\mu\xi]\lambda}^{\cdot\cdot\cdot\omega}, \end{cases}
$$

which were given by Schouten (1924.9, pp. 131–132).

To obtain the projective tensors in a systematic way, we start with the observation of T. Y. Thomas (1925. 7) that the functions

$$
(93) \qquad \Pi_{\lambda\mu}^{\nu} = \Gamma_{\lambda\mu}^{\nu} - \frac{1}{n+1}(A_{\lambda}^{\nu}\Gamma_{\mu\alpha}^{\alpha} + A_{\mu}^{\nu}\Gamma_{\lambda\alpha}^{\alpha})
$$

are unaltered by transformations (86). The $\Pi_{\lambda\mu}^{\nu}$ build up a *projective displacement*, and are subject to the identity

$$
(94) \qquad \Pi_{\lambda\alpha}^{\alpha} = 0.
$$

The paths of the projective displacements can now be made solutions of the equations

$$
(95) \qquad \frac{d^2x^{\nu}}{dp^2} + \Pi_{\lambda\mu}^{\nu}\frac{dx^{\lambda}}{dp}\frac{dx^{\mu}}{dp} = 0,
$$

where p is the projective parameter, defined but for a transformation

$$
(96) \qquad 'p = A\int\Delta^{2/(n+1)}dp + B,
$$

where $\Delta = |\partial' x^{\lambda}/\partial x^{\mu}|$ (see (3)), and where A and B are constants. This transformation involves Δ, and the same holds for that of $\Pi_{\lambda\mu}^{\nu}$:

$$
(97) \qquad \Pi_{\lambda\mu}^{\nu}\frac{\partial' x^{\omega}}{\partial x^{\nu}} = {}'\Pi_{\rho\sigma}^{\omega}\frac{\partial' x^{\rho}}{\partial x^{\lambda}}\frac{\partial' x^{\sigma}}{\partial x^{\mu}} + \frac{\partial^2{}' x^{\omega}}{\partial x^{\lambda}\partial x^{\mu}} + \frac{1}{n+1}\left(\frac{\partial' x^{\omega}}{\partial x^{\lambda}}\frac{\partial \log\Delta^{-1}}{\partial x^{\mu}} + \frac{\partial' x^{\omega}}{\partial x^{\mu}}\frac{\partial\log\Delta^{-1}}{\partial x^{\lambda}}\right).
$$

The problem of finding covariants of linear displacements may be stated as the study of the integrability conditions of the equations (9) for $\Gamma^{\nu}_{\lambda\mu}$. In the same way the problem of projective equivalence of two affine displacements may be reduced to the study of the integrability conditions of the equations (97). This has been done by Veblen-Thomas (1925. 8; 1926. 4), anticipated by J. M. Thomas (1925. 9) and T. Y. Thomas (1925. 7). They introduce *projective normal coordinates*, and use them for the deduction of projective covariants. These, however, need not have tensor character. This is only the case if $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}=0$. As the main result of this study we may consider a set of *completeness theorems* and the theorem on *projective equivalence* of two affine displacements.

For transformations with $\Delta=1$ the $\Pi^{\nu}_{\lambda\mu}$ transform like the $\Gamma^{\nu}_{\lambda\mu}$ and the p is uniquely determined. This case is called *equi-projective* (Thomas 1925. 7).

If the affine displacement can be projectively transformed so that $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}=0$, we call this case *projective-euclidean*. We then obtain from (87)

$$R_{\xi\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 2P_{[\omega\mu]}A_{\lambda}^{\nu} - 2A_{[\omega}^{\nu}P_{\mu]\lambda}. \tag{98}$$

The integrability conditions, that (91) may exist under the auxiliary condition (98), reduce to

$$\nabla_{[\omega}P_{\mu]\lambda} = 0. \tag{99}$$

When however, we apply the identity of Bianchi to (98), we obtain

$$\nabla_{[\xi}P_{\lambda\mu]} = \tfrac{1}{3}(n-2)\nabla_{[\xi}P_{\mu]\lambda}, \tag{100}$$

which for $n>2$ is identical with (99). In this case the integrability conditions of (91) are a consequence of (98). Then, as $P_{\mu\lambda}$ is defined by (91), (98) passes (see (90)) into

$$P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0. \tag{101}$$

For $n=2$ condition (98) is always satisfied. Hence we have the theorem that *an affine displacement for $n>2$ is projective-*

euclidean when and only when $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}=0$, *and for* $n=2$ *when and only when* $P_{\omega\mu}$, *defined by* (99) *satisfies* (91) (Weyl 1921. 2).

10. *Conformal Transformation of a Riemannian Manifold.* A close analogy exists between the projective transformations of affine manifolds and the conformal transformations

$$'g_{\lambda\mu} = \sigma g_{\lambda\mu}, \tag{102}$$

where σ is an arbitrary function of the x^ν, in a Riemannian manifold (102) implies

$$'g^{\lambda\mu} = \sigma^{-1}g^{\lambda\mu}. \tag{103}$$

Then (see (86)) we have

$$'\begin{Bmatrix}\lambda\mu\\ \nu\end{Bmatrix} = \begin{Bmatrix}\lambda\mu\\ \nu\end{Bmatrix} + \frac{1}{2}(\overset{\nu}{A}_\lambda s_\mu + \overset{\nu}{A}_\mu s_\lambda - g^{\nu\alpha}g_{\mu\lambda}s_\alpha), \tag{104}$$

where

$$s_\mu = \frac{\partial \log \sigma}{\partial x^\mu}\cdot \tag{105}$$

These conformal transformations leave the angle of two vectors invariant. Under them the curvature affinor $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ passes into the form

$$'R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} - g_{\omega\lambda}s_{\mu\alpha}g^{\alpha\nu} + g_{\lambda\mu}s_{\omega\alpha}g^{\alpha\nu}, \tag{106}$$

where (see (87))

$$s_{\mu\alpha} = 2\nabla_\mu s_\alpha - s_\mu s_\alpha + \tfrac{1}{2}\, s_\beta s^\beta g_{\mu\alpha}. \tag{107}$$

Equations (104) were obtained by Fubini*; equation (106) by Weyl (1918. 3).

We can ask if a conformal transformation can preserve the geodesic lines. Then, by (86) and (104) we have

$$\tfrac{1}{2}\,\overset{\nu}{A}_\lambda s_\mu + \tfrac{1}{2}\,\overset{\nu}{A}_\mu s_\lambda - \tfrac{1}{2}\, s^\nu g_{\lambda\mu} = \overset{\nu}{A}_\lambda p_\mu + \overset{\nu}{A}_\mu p_\lambda, \tag{108}$$

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* G. Fubini, *Sulla teoria degli spazi che ammettono un gruppo conforme*, Atti di Torino, vol. 38 (1903), pp. 262–276.

which cannot be fulfilled, as is easily shown by contraction with respect to ν, λ; and by transvection with $g^{\lambda\mu}$; for both operations give

$$s_\mu = \frac{2(n+1)}{n} p_\mu = -\frac{4}{n-2} p_\mu, \tag{109}$$

which has no solutions for integer n. Hence we have the theorem *that a Riemannian displacement is fully determined when we know its geodesic lines and the fundamental tensor but for a factor* (Weyl 1921. 2).

Hence projective and conformal transformations exclude each other. Eyraud (1925. 4) has shown how a general deformation of the paths can be decomposed into two deformations, one a generalization of the projective, the other of the conformal transformation.

The conformal theory of Riemannian displacements is the theory of those tensors which are unchanged by the transformation (102) (or 104). (See, however, Veblen-Thomas' remark 1926. 4.) One of these, the conformal curvature tensor, was found by Weyl (1918. 2); it is

$$C_{\omega\mu\lambda\nu} = R_{\omega\mu\lambda\nu} - \frac{2}{n-2}(g_{\omega\lambda}L_{\mu\nu} - g_{\mu\lambda}L_{\omega\nu}), \tag{110}$$

when

$$L_{\mu\lambda} = -R_{\mu\lambda} + \frac{1}{2(n-1)} R g_{\mu\lambda}. \tag{111}$$

For $n=1$, $n=2$, and $n=3$, $C_{\omega\mu\lambda\nu}$ vanishes identically. It satisfies, for $n>3$, the four identities

$$\left\{\begin{aligned} C_{\omega\mu\lambda\nu} &= -C_{\mu\omega\lambda\nu}, \\ C_{[\omega\mu\lambda]\nu} &= 0, \\ C_{\omega\mu\lambda\nu} &= -C_{\omega\mu\nu\lambda}, \\ C_{\omega\mu\lambda\nu} &= C_{\lambda\nu\omega\mu}. \end{aligned}\right. \tag{112}$$

To obtain the conformal affinor in a systematic way, we start with the remark of J. M. Thomas (1925. 12) that the functions

$$(113)\quad Z_{\lambda\mu}^{\nu} = \left\{ \begin{matrix} \lambda\mu \\ \nu \end{matrix} \right\} - \frac{1}{n}\left[A_{\lambda}^{\nu}\left\{ \begin{matrix} \mu\alpha \\ \alpha \end{matrix} \right\} + A_{\mu}^{\nu}\left\{ \begin{matrix} \lambda\alpha \\ \alpha \end{matrix} \right\} \right] + \frac{1}{n} g^{\nu\alpha} g_{\lambda\mu}\left\{ \begin{matrix} \alpha\beta \\ \beta \end{matrix} \right\}$$

are unaltered by transformations (102). The $Z_{\lambda\mu}^{\nu}$ build up a *conformal displacement*, and are subject to the identity

$$(114)\qquad Z_{\lambda\alpha}^{\alpha} = 0.$$

The paths of the conformal displacement are given by

$$(115)\qquad \frac{d^2x^\nu}{dc^2} + Z_{\lambda\ \mu}^{\ \nu}\frac{dx^\lambda}{dc}\,\frac{dx^\mu}{dc} = 0,$$

where c is the *conformal parameter*. For a discussion of this equation in the same way as (95) see J. M. Thomas (1925. 12). See also T. Y. Thomas (1925. 15; 1926. 14).

If the Riemannian displacement can be conformally transformed so that $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0$, this case is called *conformal-euclidean*. We then obtain, from (104),

$$(116)\qquad R_{\omega\mu\lambda\nu} = g_{\omega\lambda}s_{\mu\nu} - g_{\mu\lambda}s_{\omega\nu}.$$

The integrability conditions that (116) may exist under the auxiliary condition (107), reduce to

$$(117)\qquad \nabla_{[\omega}s_{\mu]\lambda} = 0.$$

When, however, we apply the identity of Bianchi to (116), we obtain

$$(n-3)\nabla_{[\omega}s_{\mu]\lambda} = 0,$$

which, for $n>3$, is identical with (117). Then, as $s_{\mu\lambda}$ is defined by (107), (116) passes into (see (110))

$$C_{\omega\mu\lambda\nu} = 0.$$

For $n=3$, condition (116) is always satisfied. We thus have (see the end of §9) *that a Riemannian displacement for* $n>3$ *is conformal-euclidean when and only when* $C_{\omega\mu\lambda\nu}=0$, *and for* $n=3$, *when and only when* $s_{\omega\mu}$, *defined by* (116), *satisfies* (117) (Schouten 1921. 7; for $n=3$ Cotton,* Finzi†).

11. *The Point of View of Point Displacement.* The ideas expressed above all start with the consideration of *vectors* in the E_n at one point P of the X_n that are compared to *vectors* in an E_n at another point P' at infinitesimal distance. Cartan has made the remark that this is not the only way to proceed. We can also start with the *points* in the E_n at a point P of X_n and define our displacement as a law that enables us to compare the *points* at P with those at P'. In this manner, Cartan first derived in a new way the asymmetric affine displacement (1922. 8), and later on could come to generalizations of projective and conformal geometry. We limit ourselves here to an exposition of Cartan's ideas for an affine displacement, and only give some outlines of his further generalizations. Consider, for this (Cartan 1923. 21, the notation is chiefly based on Schouten 1924. 8 and 1926. 12), the E_n of the dx^ν at a point P of the X_n and the E_n' at P'. Then the formulas give a law of comparison of the *free* vectors E_n and E_n'. We get a law of comparison of the *points* at E_n and E_n' if we also know how a definite point of E_n passes into a definite point of E_n'. Let us now first obtain in this new way the affine displacements. Introduce in the E_n *homogeneous* coordinates. Every set u^α, $\alpha=0, 1, \cdots, n+1$, abbreviated $\bar{u}$, determines a *point* by its proportion. Then a fundamental $(n+1)$-cell is defined by $n+1$ points $\bar{u}_0, \bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$. Let $\bar{u}_0$ be P itself, and let $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ be the points at infinity at n oblique axes issuing from P. If

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* E. Cotton, *Sur les variétés à trois dimensions*, Annales de Toulouse, (2), vol. 1 (1899), pp. 385–438; also Thèse, Paris, 1899, 61 pp.

† A. Finzi, *Le ipersuperficie a tre dimensioni che si possono rappresentare conformemente sullo spazio euclideo*, Atti, Istituto Veneto, (8), vol. 5(=62), (1902–03), pp. 1049–1062.

we now assume an affine correspondence between E_n at P and E_n' at P', we get our linear connection defined by

$$(118) \qquad \delta \bar{u}_\lambda = \overset{\kappa}{\Gamma}_{\lambda\mu} \bar{u}_\kappa dx^\mu, \quad (\lambda, \mu, \kappa = 1, \cdots, n).$$

If we now fix the correspondence of P to its image in E' by the equation

$$(119) \qquad \delta \bar{u}_0 = - \bar{u}_\mu dx^\mu,$$

a linear connection is determined:

$$(120) \qquad \begin{cases} \nabla_\mu v^\alpha = \dfrac{\partial v^\alpha}{\partial x^\mu} + \overset{\alpha}{\Lambda}_{\beta\mu} v^\beta, \\ \nabla_\mu w_\alpha = \dfrac{\partial w_\alpha}{\partial x^\mu} - \overset{\beta}{\Lambda}_{\alpha\mu} w_\beta, \qquad (\alpha, \beta = 0, 1, \cdots, n), \end{cases}$$

where

$$(121) \qquad \begin{cases} \overset{\nu}{\Lambda}_{\lambda\mu} = \overset{\nu}{\Gamma}_{\lambda\mu}, \\ -\overset{\nu}{\Lambda}_{\lambda 0} = \overset{\nu}{A}_{\mu} \begin{cases} =1, & \nu = \mu, \\ =0, & \nu \neq \mu, \end{cases} \\ \overset{0}{\Lambda}_{\alpha\mu} = 0. \qquad (\lambda, \mu, \nu = 1, \cdots, n). \end{cases}$$

A special case of (119) was treated by Hlavatý (1924. 21). These covariant differentials are mixed quantities, because the differentials dx^μ and the vectors $\bar{u}$ belong to different manifolds. In X_n, however, a displacement is not fixed. But if we assume

$$(122) \qquad 2\nabla_{[\mu} w_{\lambda]} = \frac{\partial w_\lambda}{\partial x^\mu} - \frac{\partial w_\mu}{\partial x^\lambda}$$

which means that this displacement is at least symmetric, we can define $\nabla_{[\omega}\nabla_{\mu]} v^\nu$ and $\nabla_{[\omega}\nabla_{\mu]} v_\lambda$ and obtain a curvature quantity

$$(123) \qquad \mathfrak{A}_{\omega\mu\alpha}^{\cdot\cdot\cdot\beta} = \frac{\partial}{\partial x^\mu} \overset{\beta}{\Lambda}_{\alpha\omega} - \frac{\partial}{\partial x^\omega} \overset{\beta}{\Lambda}_{\alpha\mu} + \overset{\beta}{\Lambda}_{\delta\mu} \overset{\delta}{\Lambda}_{\alpha\mu} - \overset{\beta}{\Lambda}_{\delta\omega} \overset{\delta}{\Lambda}_{\alpha\mu},$$
$$(\omega, \mu = 1, \cdots, n;\ \alpha, \beta, \delta = 0, 1, \cdots, n).$$

For this tensor we have the first identity and Bianchi's identity

$$(124)\qquad \begin{cases} \mathfrak{A}_{\omega\mu\alpha}^{\cdot\cdot\cdot\beta} = -\mathfrak{A}_{\mu\omega\alpha}^{\cdot\cdot\cdot\beta} \\ \nabla_{[\xi}\mathfrak{A}_{\omega\mu]\alpha}^{\cdot\cdot\cdot\beta} = 0. \end{cases}$$

$\mathfrak{A}_{\omega\mu\alpha}^{\cdot\cdot\cdot\beta}$ can be derived from $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$ and $S_{\mu\lambda}^{\cdot\cdot\nu}$. We have

$$(125)\qquad \begin{cases} \mathfrak{A}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}\ ; & \mathfrak{A}_{\omega\mu\lambda}^{\cdot\cdot\cdot 0} = 0 \\ \mathfrak{A}_{\omega\mu 0}^{\cdot\cdot\cdot\nu} = -2S_{\omega\mu}^{\cdot\cdot\nu}\ ; & \mathfrak{A}_{\omega\mu 0}^{\cdot\cdot\cdot 0} = 0. \end{cases}$$

By means of the formula $Dv^\beta = f^{\omega\mu}\, d\sigma\ \mathfrak{A}_{\omega\mu\alpha}^{\cdot\cdot\cdot\beta} v^\alpha$, we see that a symmetric displacement is characterized by the fact that $\bar{u}_0$ returns in its own place after a transport along a closed circuit. This is another statement of the fact expressed by formula (24), that in the non-symmetric case the manifold is "infinitely crinkled."

If $R_{\omega\mu\alpha}^{\cdot\cdot\cdot\gamma} = 0$, we get an E_n. In this case the displacement passes *into the correspondence of all points of the E_n into themselves.*

This last remark leads toward a generalization for the projective and conformal cases, which can now be treated in a way analogous to the preceding paragraph (Cartan 1923. 20; 1924. 4; 1924. 3). The difference is that the E_n at a point P of an X_n must be considered as a projective or conformal space. By the introduction of homogeneous coordinates, projective space is equivalent to affine space of $n+1$ dimensions; by the introduction of tetracyclical, pentaspherical, etc., coordinates, conformal space is equivalent to affine space of $n+2$ dimensions. We thus come to the following generalization: Associate to every point P of an X_n an E_{n+k}, $k=0, 1, 2, \cdots$ and generalize the method of the preceding paragraph to this case. We then have three kinds of quantities:

(a) Quantities in X_n, defined for the E_n of the dx^ν.

(b) Quantities in the E_{n+k} defined in every E_{n+k} with respect to a system of $n+k$ covariant and $n+k$ contravariant fundamental vectors.

(c) Quantities obtained by summing and multiplying the quantities of first and second kind.

We obtain a displacement if we determine how the quantities of the second kind of the E_{n+k} at P correspond to those of the E_{n+k} at a neighboring point P'. We thus obtain a method of treating the results of point displacements by the methods of vector displacement, and find a kind of geometry previously introduced, at least in general, by König (1919. 1; 1920. 1).

Cartan, using his point displacement, now proceeds in an analogous way as in the affine case. He deduces a curvature tensor for the projective and the conformal displacement

$$\mathfrak{P}_{\omega\mu a}^{\cdot\cdot\cdot c}; \qquad \begin{aligned} \omega,\mu &= 1,\cdots,n \text{ in } X_n;\\ a,c &= 0,1,\cdots,n \text{ in } E_{n+1}, \end{aligned} \tag{126}$$

and

$$\mathfrak{C}_{\omega\mu a c}; \qquad \begin{aligned} \omega,\mu &= 1,\cdots,n \text{ in } X_n;\\ a,c &= 0,1,\cdots,n+1 \text{ in } E_{n+2}. \end{aligned} \tag{127}$$

The relation between point displacements of Cartan and the vector displacements are given by the theorems:

If in an X_n an affine displacement is fixed but for transformations that leave the paths invariant, there is one and only one symmetric projective point connection with the same geodesic lines, for which

$$\sum_k \mathfrak{P}_{kli}^{\cdot\cdot\cdot k} = 0; \qquad k,l = 1,\cdots,n.$$

Moreover, we have

$$\mathfrak{P}_{kli}^{\cdot\cdot\cdot m} = P_{kli}^{\cdot\cdot\cdot m}.$$

The special coordinate system in X_n with respect to which their components are taken is one for which the given affine displacement preserves volume (Cartan 1924. 4, Schouten 1924. 8).

If in an X_n, $n \geqq 3$, *Riemann displacement is fixed but for conformal displacements, there is one and only one symmetric conformal point connection with the orthogonality fixed by* $g_{\mu\nu}$, *for which* $\sum_k \mathfrak{C}_{klik}=0$; $k, l=1, \cdots, n$. Moreover, $\mathfrak{C}_{klim}=C_{klim}$.

The coordinate system in X_n with respect to which their components are taken is an orthogonal system (Cartan 1923. 20). Cartan's results are discussed from the point of view of vector displacement by Schouten (1926. 12), where also some general remarks on their relation toward the group program of Felix Klein's Erlanger address can be found. A good introduction to Cartan's conceptions is given by his papers of a more general character (1924. 5; 1925. 1).

12. *Manifolds Immersed in Manifolds with a Linear Displacement.* An important reason for the study of X_m, $m<n$, in an X_n with a linear displacement was the desire to have a treatment of affine differential geometry, developed by Blaschke and others,* on the basis of linear displacement. König (1919. 1) pointed out the way, and showed that this affine geometry was the geometry of a linear displacement with

$$(128) \qquad C_{\lambda\mu}^{\cdot\cdot\nu} = 0, \quad \Gamma_{\lambda\mu}^{\nu} = \Gamma_{\mu\lambda}^{\nu}, \quad R_{\alpha\mu\lambda}^{\cdot\cdot\cdot\nu} = 0.$$

Schouten (1923. 7; 1924. 9) took up König's suggestion and gave an elaborate study of this subject. Cartan (1924. 22; 1924. 1; 1923. 21) studied the same subject independently of König. Schouten starts with an X_n and a displacement for which $C_{\lambda\mu}^{\cdot\cdot\nu}=0$, $\Gamma_{\lambda\mu}^{\nu}=\Gamma_{\mu\lambda}^{\nu}$. Consider now an X_{n-1} immersed in the X_n, and a point P of X_{n-1}. Let v^ν be a vector field in X_n; if a vector v^ν is not situated in the tangential E_{n-1} at P, we cannot speak without further assumption of the v^ν-*component* of v^ν in E_{n-1}. This is only possible after the

* See W. Blaschke, *Vorlesungen über Differentialgeometrie*, II, *Affine Differentialgeometrie*, Berlin, Springer, 1923.

introduction of a special direction at P not situated in E_{n-1}, the *pseudonormal* direction. As a covariant vector can be represented by a set of two parallel E_{n-1}, we see that a covariant vector determines, by its intersection with the tangential E_{n-1}, a covariant vector in this E_{n-1} (representable by two parallel E_{n-2}), without any further assumption. After the introduction of a pseudonormal direction, however, it is possible to associate to a covariant vector field in X_{n-1} also a covariant vector field in X_n.

In the same way, we can consider tensors, especially the covariant derivatives of scalars and vectors. Let B_β^α be the unit tensor of the X_{n-1}, t_λ the tangential covariant vector of the X_{n-1}, and n^ν a vector in the pseudonormal direction. If we now assume $t_\lambda n^\lambda = 1$, which does not fix the scale of measure of t_λ, we have between B^ν and the unit tensor A_λ^ν of X_n the relation

$$B_\lambda^{\nu} = A_\lambda^{\nu} \ t_\lambda n^\nu. \tag{129}$$

We now define, ∇_μ being the differential operator with respect to the X_n, a linear displacement in the X_{n-1} by means of the equations

$$\left\{\begin{aligned} {}'\nabla_\mu v^\nu &= B_\mu^{\alpha} B_\beta^{\nu} \nabla_\alpha v^\beta \\ {}'\nabla_\mu w_\lambda &= B_\mu^{\alpha} B_\lambda^{\beta} \nabla_\alpha w_\beta . \end{aligned}\right. \tag{130}$$

Even before introducing a normal vector, we may define the derivative of a scalar field

$${}'\nabla_\mu p = B_\mu^{\alpha} \nabla_\alpha p. \tag{131}$$

The displacement thus defined is also affine (see (17) and (27), since we have

$$\left\{\begin{aligned} {}'\nabla_\mu B_\lambda^{\nu} &= B_\mu^{\alpha} B_\lambda^{\beta} B_\delta \nabla_\alpha^{\nu} B_\beta^{\delta} = - B_\mu^{\nu} B_\lambda^{\beta} B_\lambda^{\nu} \nabla_\alpha t_\beta n^\delta = 0, \\ {}'\nabla_{[\omega} {}'\nabla_{\mu]} p &= {}'\nabla_{[\omega} (B_{\mu]}^{\alpha} \nabla_\alpha p) = B_\mu^{\alpha} \nabla_{[\omega}' \nabla_{\alpha]} p \\ &= B_\mu^{\alpha} B_\omega^{\beta} \nabla_{[\beta} \nabla_{\alpha]} p = 0, \end{aligned}\right. \tag{132}$$

We thus have the fundamental theorem :

If we define at every point of the X_{n-1} a pseudonormal direction, we can define an affine displacement in the X_{n-1} in *such a way that the covariant derivative of a quantity in* X_{n-1} *is the* X_{n-1}*-component of the covariant derivative with respect to the* X_n.

The displacement in X_{n-1} is called *induced*. We now can induce the following tensor:

$$h_{\mu\lambda} = B_{\mu}^{\alpha} B_{\lambda}^{\beta} \nabla_{\alpha} t_{\beta}, \tag{133}$$

which, by choice of the part of t_β outside of the X_{n-1}, can be made symmetric. Change of the scale of measure for t_λ of the form $'t_\lambda = \sigma t_\lambda$, changes the tensor $h_{\mu\lambda}$ conformally:

$$'h_{\mu\lambda} = \sigma B_{\mu}^{\alpha} B_{\lambda}^{\beta} \nabla_{\alpha} t_{\beta} + B_{\mu}^{\alpha} B_{\lambda}^{\beta} t_{\beta} \nabla_{\alpha} \sigma = \sigma h_{\mu\lambda}. \tag{134}$$

This tensor is perfectly determined after choice of the scale of measure t_λ. In this case a Riemannian displacement is fixed in the X_{n-1}, if $h_{\lambda\mu}$ has the rank $n-1$, and then we take $h_{\lambda\mu}$ as fundamental tensor.

If $P_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0$ in the X_n we can show that

$$T_{\lambda\mu\nu} = T_{\lambda\mu}^{\cdot\cdot\alpha} h_{\alpha\nu} \tag{135}$$

(see (38)) is symmetric, and also that (see (32))

$$T_{\lambda\mu\nu} = \tfrac{1}{2} Q_{\lambda\mu\nu}, \quad h^{\mu\nu} Q_{\lambda\mu\nu} = 0.$$

This condition of apolarity (135) is one of the signs that for the case of ordinary affine differential geometry we have in $h_{\mu\lambda}$ and $Q_{\lambda\mu\nu}$, the so-called first and second fundamental tensors. This geometry is obtained, when the X_n becomes an E_n, through the condition $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} = 0$. The theory of X_{n-1} in X_n and also X_k in X_n $(k < n-1)$ can be developed on this basis. We can obtain, for example, the generalizations of the formulas of Gauss and Codazzi for these cases.

The case of $\Gamma_{\lambda\mu}^{\nu} \neq \Gamma_{\mu\lambda}^{\nu}$ for X_k in X_n was treated by Hlavatý (1926. 13).

Another treatment of these geometries was given by Cartan (1923. 21, vol. 40, p. 403; Cartan has many results not mentioned here). Moreover, Schouten (1924. 9) developed the theory of X_k in X_n for a Weyl geometry.

Frenet formulas for curves in a general linear displacement and a Weyl displacement, but with fixed $g_{\lambda\mu}$, were found by Juvet (1921. 5 ; 1921. 6 ; 1924. 13 ; 1925. 20). Schouten gave the general formulas (1924. 9); see also Juvet (1924. 14).

Eisenhart discussed parallel vector fields in an affine displacement (1922. 4). He calls the vectors of a field *parallel* to one another, if a scalar α exists, so that we have, independent of the direction of displacement dx^ν,

$$dx^\lambda \nabla_\lambda \alpha v^\nu = 0.$$

13. *Linear Displacements and Continuous Groups.* If we consider the ∞^r transformations of an r-parametric continuous group in an X_n as points of an X_r, we obtain the *group-manifold,* in which the parameters are coordinates. Let the transformations be

$$(136)\quad 'x^\nu = 'x^\nu(x^1, \cdots, x^n, a^1, \cdots, a^r), \quad (\nu = 1, \cdots, n) ;$$

then the a^λ, $\lambda = 1, \cdots, r$, are the coordinates of X_r. The transformation (136) may be denoted by T_a.

If T_a corresponds with the point a^ν and T_{a+da} with the point $a^\nu + da^\nu$, then the infinitesimal transformation $T_{a+da}T_a^{-1}$ corresponds to the line element da^ν. The line element db^ν at point b^ν is characterized by $T_{b+db}T_b^{-1}$. This correspondence of both transformations can be expressed by

$$T_{b+db}T_b^{-1} = T_{a+da}T_a^{-1}.$$

This defines a linear displacement, through which a vector at a point corresponds to one and only one vector at another point. Hence $R_{\omega\mu\lambda}^{\cdot\cdot\cdot\,\nu} = 0$. It can be shown, however, that $\Gamma_{\lambda\mu}^\nu \neq \Gamma_{\mu\lambda}$. Another displacement with these two properties can be defined by

$$T_b^{-1}T_{b+db} = T_a^{-1}T_{a+da}.$$

The study of simple and semi-simple groups* under this point of view was started by Cartan-Schouten (1925. 3). Related to these investigations is a paper of Eisenhart (1925. 10).

* First introduced by E. Cartan, *La structure des groupes de transformations continus et la théorie du trièdre mobile,* Bulletin des Sciences Mathématiques, (2), vol. 34 (1910), pp. 250–283.

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