It is therefore wholly to its credit that Bell's book is so largely abstract. In this way he has secured the utmost generality and hence insured permanency to his work. There is also the following important immediate gain. After constructing abstract theories for various classes of problems, he was in a position to decide if those theories are distinct or are (abstractly) identical. In the latter case we have not merely the often astonishing conclusion that two quite different theories are really equivalent, but the great advantage of being able to apply in a particular case whichever of the two theories is best adapted to it.

The book is however not abstract for the sake of being abstract, but because it gives the essence of a vast array of concrete results traced to their true sources and easily deduced from these few sources. It is not to be overlooked that the author has a minute first-hand acquaintance with the vast array of concrete facts in the theory of numbers. It would require many volumes to expound them and the result would bewilder the reader. How much better to have a brief book which epitomizes all these facts under a few abstract theories. This original and scholarly book is an honor to American mathematics.

L. E. DICKSON

FORSYTH ON CALCULUS OF VARIATIONS

Calculus of Variations. By A. R. Forsyth. Cambridge University Press, 1927. xxii+656 pp.

Weierstrass made three very important contributions to the theory of the calculus of variations. Earlier writers had deduced necessary conditions on a minimizing arc $y=y(x)$ $(x_1\leqq x\leqq x_2)$ by comparing the value which such an arc gives to the integral to be minimized with the value given by neighboring arcs of the form $y=y(x)+\delta y(x)$, where $\delta y=a\eta(x)$. By the use of variations δy of another type, Weierstrass deduced a new necessary condition. He also introduced the parametric representation of curves, $x=x(t)$, $y=y(t)$, into the theory. This was important from the standpoint of the older writers because it enables one to vary impartially either of the variables x or y. The effort to do this had caused considerable difficulty and misunderstanding in the earlier theory of the non-parametric case. The parametric theory is even more valuable, however, because it removes the geometric restrictions on the form of curves which are imposed by the non-parametric representation $y=y(x)$, a removal which is necessary for the complete investigation of many geometric problems. Finally Weierstrass formulated clearly the problems which he studied, distinguished clearly between conditions which are necessary for a minimum and those which are sufficient, and devised an ingenious sufficiency proof which under certain circumstances established the minimizing property of an arc $y=y(x)$ as compared with all other neighboring arcs $y=y(x)+\delta y(x)$, irrespective of the form of the variation δy.

In the book here reviewed Professor Forsyth shows that he has been influenced by the first two of the contributions which have just been mentioned as being due to Weierstrass, the so-called Weierstrassian necessary condition and the introduction of the parametric representation of curves. The method which he uses is that of expansion in series of powers of the variations of the functions defining the curves, the various necessary conditions being deduced by the study and transformation of the first and second variations of the integral to be minimized. The cases which he considers are problems in the plane for integrals whose integrands contain first, or first and second, derivatives of the functions defining the arc, in both non-parametric and parametric forms, the analogous problems in three-space, isoperimetric problems, double integrals containing first or second partial derivatives and isoperimetric problems involving double integrals, and triple integrals involving first partial derivatives.

In the first paragraph of his introduction Professor Forsyth says: "The range of Mathematical Analysis, usually known as the Calculus of Variations, deals with one of the earliest problems of ordinary experience. The requirement was, and is, to obtain the most profitable result from imperfectly postulated data; and the data may possibly be subjected to conditions, which likewise are imperfectly postulated. When data and conditions are expressed in analytical form, the necessary mathematical calculations can not be effected directly, because of some essential deficiency in the information." It seems to me that this rather vague type of language, which is also used in other places in the book, gives a very vague impression of the theory of the problems of the calculus of variations. The data of these problems can be postulated with accuracy, and the properties deduced for their solutions will in general characterize those solutions completely.

There is a strong tendency among modern writers to avoid as far as ossible the elaborate transformations of the second variation which characterized earlier theories. Legendre's condition, which was originally deduced from the second variation, is in many cases an immediate consequence of the condition of Weierstrass which can be proved without using the second variation at all. Jacobi's condition is a consequence of the beautiful envelope theorem of Darboux, Zermelo, and Kneser, which is a generalization of the string property of the evolute of a curve, or it may be deduced from the second variation by a method which requires almost no manipulation.* For these reasons I always regret to see the elaborate transformation theory of the second variation given such a prominent place as Professor Forsyth has given it in his book.

In a paper read at the Toronto Congress, and which has recently been printed for publication in the Proceedings of the Congress, I showed that the result of the transformation which Clebsch devised for the second variation, in the very general case of the Lagrange problem, is obtained as an immediate consequence of a theorem of Weierstrass expressing the difference between the values of an integral on two neighboring curves as an integral of its E-function. So much effort has been spent on these

* See the references in the footnote, this Bulletin, vol. 32 (1926), p. 392; for the parametric case, see the Transactions of this Society, vol. 17 (1916), p. 195.

transformations that it seemed to me worth while to try to coordinate the transformation theory with other well known results in the calculus of variations. If the second variation must be transformed it is desirable to do it in some such way as this.

It has frequently surprised me to find how much effort the early writers on the calculus of variations spent on the theory of problems for which the integrand functions contain higher derivatives than the first. There are very few special problems of this sort in the literature. Furthermore the theory of such problems is a special case of the theory of the Lagrange problem which has a much wider application. Professor Forsyth discusses only relatively simple forms of the problem of Lagrange but he studies at length problems in the plane and in three-space for which second derivatives occur in the integrand. The theory of the second variation is of course very complicated for these problems. The parametric theory for such cases could doubtless be simplified by the introduction of invariants under parametric transformation in place of higher derivatives in the integrand, as has been done by Radon.*

The last two hundred pages of the book are devoted to double and triple integrals, with applications to minimal surfaces, and the usual necessary conditions are deduced from the first and second variations. The method of transforming the second variation of parametric problems is a modification due to Professor Forsyth of that used by Kobb. The two final chapters are devoted to double integrals whose integrands contain second derivatives, and to triple integrals, and the theory of the second variation for these cases is more completely treated than I have seen it done elsewhere.

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* *Über das Minimum des Integrals* $\int_{s_0}^{s_1} f(x, y, \theta, \kappa)ds$, Wiener Sitzungsberichte, vol. 119 (1910), p. 1257.