CONCERNING COVARIANTS OF THE RATIONAL PLANE QUARTIC CURVE WITH COMPOUND SINGULARITIES*

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1. *Introduction.* After investigating the invariants of the rational plane quartic curve R_2^4 with compound singularities† it is of interest to see what effect such singularities have on covariants of the curve. This paper observes the effect of the tacnode and the ramphoid cusp upon associated curves and sets of covariant parameters.

2. *Covariant Forms of the Curve with a Tacnode.* To examine these forms we let $t = \pm a$ be the tacnodal parameters and t_1 and t_2 the contacts of the distinct double line of the curve with an acnode. This may be done if 0 and ∞ are the contacts of the two tangents to the curve from the tacnode.‡ These tangents and the line through their contacts give the very neat representation

$$(1) \qquad \begin{cases} x_0 = t^4 - a^2t^2, \\ x_1 = t^2 - a^2, \\ x_2 = t^3 + st^2 + \sigma_2 t, \end{cases}$$

where $\sigma_1 = t_1 + t_2$, $\sigma_2 = t_1t_2$, and $s = [a^2\sigma_1^2 + (a^2+\sigma_2)^2]/(2a^2\sigma_1)$.

The pencil of conics§

$$(2) \qquad g_2 - \lambda K = 0,$$

where g_2 is the envelope of lines which cut the curve in self-apolar sets of points and K is the locus of the vertices

* Presented to the Society, April 6, 1928.

† Neelley, *Compound singularities of the rational plane quartic curve*, American Journal of Mathematics, vol. 49 (1927), pp. 389–400.

‡ Neelley, loc. cit., p. 394.

§ J. E. Rowe, *Covariants and invariants of the rational plane quartic*, Transactions of this Society, vol. 12 (1911), pp. 298–299.

of the flex-triangles of the first osculants of R_2^4, is not affected by the singularity. It has one member which passes through the tacnode and that is the conic on the contacts of the double lines. This conic has the tacnodal line

$$x_0 - a^2x_1 = 0 \tag{3}$$

as its tangent at the tacnode which is to be expected as several double line contacts coincide at a tacnode.* The tacnode does not affect the covariant line on the points q_i† nor the line which is the locus of a point such that tangents from it to R_2^4 are apolar to the flexes.†

Next we examine the combinants of two line sections (ξx) and (ηx) of the curve (1).‡ These combinants also seem to be independent of the tacnode except C which has $x_0-a^2x_1$ as a squared factor. This was expected as C gives the double lines of R_2^4. The fundamental involution§ determined by two binary quartics does not give any special information as none of the forms which make up the complete system|| of these two quartics vanish identically.

3. *Covariant Forms of the Curve with a Ramphoid Cusp.* This singularity is a special form of the tacnode but it affects the covariants very strikingly. Suppose $t=0$ at the cusp and $t=\infty$ at a distinct real flex. There is always such a flex as three flexes coincide at the cusp.¶ Then the cuspidal line, the flex line and the join of cusp and flex give

$$\left\{\begin{aligned} x_0 &= at^4, \\ x_1 &= 4bt^3 + 6ct^2, \\ x_2 &= 4dt + e. \end{aligned}\right. \tag{4}$$

Two lines (ξx) and (ηx) cut out the line sections

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* Neelley, loc. cit., p. 395.

† Rowe, Transactions of this Society, vol. 13, p. 390.

‡ Salmon, *Modern Higher Algebra*, 4th ed., *p*. 296.

§ Meyer, *Apolarität und rationale Kurven*, p. 9.

|| Salmon, loc. cit., p. 224.

¶ Neelley, loc. cit., p. 390.

$$(5)\qquad \begin{cases} U \equiv a\xi_0 t^4 + 4b\xi_1 t^3 + 6c\xi_1 t^2 + 4d\xi_2 t + e\xi_2 = 0, \\ V \equiv a\eta_0 t^4 + 4b\eta_1 t^3 + 6c\eta_1 t^2 + 4d\eta_2 t + e\eta_2 = 0. \end{cases}$$

The pencil of conics considered in §2 is derived from combinants of $U+\mu V$ by means of the rate of exchange

$$\begin{vmatrix} (a\xi)\,(b\xi) \\ (a\eta)\,(b\eta) \end{vmatrix} = \begin{vmatrix} a & & \\ & b & \\ & & x \end{vmatrix}^{*},$$

and so on. Here g_2 has the equation

$$(6)\qquad 16b^2d^2x_0^2 + a^2e^2x_1^2 + 12ac^2ex_0x_2 + 8abdex_0x_1 = 0,$$

and K the equation

$$(7)\qquad b^2d^2x_0^2 + acd^2x_0x_1 + ac^2ex_0x_2 = 0.$$

Obviously g_2 is on the cusp (0, 0, 1) and has the line $x_0=0$ as its tangent at that point. Also K is degenerate with $x_0=0$ as one of its lines. So g_2 and K have contact at the cusp and the pencil is composed of conics having contact with R_2^4 at the cusp, all of the conics cutting out at least four coincident points of the quartic at that point. So the octavics cut out by the pencil of conics have four-fold roots. Then if $\lambda=12$ we have another degenerate member of the pencil. This is the Stahl conic N† which is the envelope of the flex lines of the cubic osculants of R_2^4. Also since the conic on the flexes and that on the contacts of the double lines are members of this pencil, we have means of proving the following known theorem.‡

THEOREM 1. *Three flexes and one of the points q_i coincide at a ramphoid cusp and two double lines coincide there.*

The pencil of covariant octavics cut out by the pencil is

$$(8)\qquad \begin{aligned}(16-\lambda)&b^2d^2t^8 + 4bd(8be - cd\lambda)t^7 \\ &+2(8b^2e^2 + 24bcde - 3c^2d^2\lambda)t^6 \\ &+ 4ce(12be + 3cd - cd\lambda)t^5 + c^2e^2(48-\lambda)t^4 = 0,\end{aligned}$$

* Rowe, loc. cit., p. 296.

† Stahl, *Ueber die rationale ebene Curve vierter Ordnung*, Journal für Mathematik, vol. 101 (1886), pp. 300–325.

‡ Neelley, loc. cit., p. 390 and p. 395; Wieleitner, *Theorie der ebenen algebraischen Kurven höherer Ordnung*, 1905, pp. 280–281.

which has several covariant forms that vanish identically due to the four-fold root.* When $\lambda = 48$, we have the conic on the contacts of the double lines and the octavic it cuts out has a five-fold root, and so a $C_{2,10}$† vanishes also.

The line on the points q_i is

$$(9) \qquad 8(bde + 3cd^2)x_0 - ae^2x_1 = 0,$$

and the locus of a point such that tangents from it to the curve are apolar to the flexes is the line

$$(10) \qquad 2(bde - 6cd^2)x_0 - ae^2x_1 = 0.$$

So these lines cut out a pencil of covariant quartics

$$(11) \quad [bde(1 - 5\lambda) - 6cd^2(1 + 3\lambda)]t^4 - 2be^2t^3 - 3ce^2t^2 = 0,$$

with at least a double root, and again we have covariant forms which vanish. One member is the square of a quadratic and so it has a $C_{3,6}$‡ which vanishes. This quartic is cut out by the distinct double line on the cusp. The cuspidal line gives a member of the pencil with a four-fold root and so several covariant forms vanish.‡ The bicombinant conic on the nodes and the points q_i also has contact with R_2^4 at the cusp and so cuts out another octavic with a four-fold root.

The Salmon combinant C§ of U and V gives the double lines of R_2^4 with the equation

$$\begin{aligned}(12) \quad 8b^4d^4x_0^4 &+ 4(6ab^2cd^4 - ab^3d^3e)x_0^3x_1 \\ &+ 2(ab^4e^3 + 9ab^2c^2d^2e - 6ab^3cde^2)x_0^3x_2 \\ &+ 6(3a^2c^2d^4 - a^2bcd^3e)x_0^2x_1^2 \\ &+ 3(a^2b^2ce^3 - 6a^2bc^2de^2 + 9a^2c^3d^2e)x_0^2x_1x_2 = 0,\end{aligned}$$

which gives the following theorem.

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* Tracey and Moore, *Covariant conditions for multiple roots of binary forms*, soon to appear in the American Journal of Mathematics.

† Cayley, Collected Mathematical Papers, vol. II, p. 316.

‡ Cayley, loc. cit., p. 555.

§ Salmon, loc. cit., p. 296.

THEOREM 2. *Two double lines coincide at a ramphoid cusp and a third double line is on the cusp, or five contacts of double lines coincide at a ramphoid cusp.*

The combinant D gives the flex lines, and their equation is of the form

$$acx_0^3 x_2 \cdot Q = 0, \tag{13}$$

where Q is a quadratic in x_0, x_1, x_2. This form proves the following theorem.

THEOREM 3. *Three flex lines unite to form the cuspidal line at a ramphoid cusp.*

We also find x_0 to be a factor of the combinant E which is the envelope of lines which cut R_2^4 in catalectic quartics.

This close connection between the cusp and covariant forms is very nicely brought out by considering the invariants of a pencil of line sections on the cusp. Suppose $(\xi x)=0$ is the cusp line at $t=0$ and $(\eta x)=0$ is on the cusp and also has contact with R_2^4 at $t=\infty$. Then

$$(a\xi)(\alpha t)^4 \equiv at^4, \quad (b\eta)(\beta t)^4 \equiv 6ct^2,$$

and so*

$$I_2 = I_2' = I_4 = I_6 = 0.$$

But I_2 is the conic K and I_2' is the conic N so these meet at the cusp. Also I_4 is R_2^4 itself and I_6 gives the flex lines. So a cusp can only be at an intersection of K and N with R_2^4 where a flex line cuts the curve.

The significant thing we note concerning these covariant forms is that they cut out covariant sets of parameter values which are given by binary forms with multiple roots. This multiplicity is such as to be characteristic of the curve with a ramphoid cusp.

Consideration of the group of covariant forms which constitute the complete system of the two binary quartics giving the fundamental involution of the curve shows that one

* Morley invariants of two binary quartics.

member of the system vanishes identically. This form is the quadratic form obtained by operating with the quartic

$$(\beta t)^4 \equiv 12bt^2 + 12ct$$

on the sextic $12(2, H_2)$, where 2 is the second quartic $(\gamma t)^4 \equiv 4dt^4 + 4et^3$ and H_2 is its Hessian. It seems that *the identical vanishing of this covariant form is a necessary and sufficient condition that R_2^4 should have a ramphoid cusp.* We arrive at this conclusion because this covariant does not vanish for any other singularity and its vanishing is independent of the reference scheme giving the curve.

This investigation gives one other theorem that is not known. We noted that the conic K is degenerate when the curve has a ramphoid cusp. Previously* it was pointed out that this was also true when the curve had one or more undulations. So it seems that K is degenerate if there is a line of R_2^4 which cuts out four points of the curve with the same parameter value. Besides the undulations and the ramphoid cusp, the only type of curve where this is true is the limiting form of the oscnode when one loop is drawn up so as to become a point: Any line on this point cuts out three points and the oscnodal line cuts out four. The conic K for this form is the square of the oscnodal line. So we do have the following thorem.

THEOREM 4. *The Stahl conic K degenerates if the quartic curve has a line which cuts out four coincident points with the same parameter value and this line is one of the lines of the conic. The types of quartics with two such lines, the curve with two undulations or an undulation and a ramphoid cusp, have K given by the product of the two lines. When there are three such lines the equation of K vanishes identically.*

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* Neelley, *A note on the rational plane quartic curve with cusps or undulations*, this Bulletin, vol. 34 (1928), pp. 639–645.