MAXIME BÔCHER*

BY WILLIAM F. OSGOOD

The twelfth of last September marked a decade since Bôcher passed away. He retained the full vigor of his mind till shortly before the end, but the last months of his life were clouded by a wasting disease, to which he finally succumbed.

Maxime Bôcher was born in Boston, August 28, 1867. He prepared for college at the Cambridge Latin School and graduated from Harvard in the Class of '88. He then went to Göttingen, where he spent six semesters and took the degree of Doctor of Philosophy in 1891. He was appointed instructor in mathematics at Harvard and entered on his duties in the fall of that year.

To understand Bôcher as a teacher, we must first look at mathematics in Cambridge in the late eighties. The traditional regard in which mathematics had always been held in New England was enhanced by the influence of Benjamin Peirce, a rugged figure in the scientific life of his time. With Byerly and B. O. Peirce came not only more effective instruction, but a systematic organization of basal courses in the calculus, in geometry, in mechanics, and in mathematical physics. Cole had just brought from Leipzig Klein's geometric treatment of the theory of functions of a complex variable, and his enthusiasm for the mathematics that had been revealed to him in Klein's seminar—for algebra and geometry, even more than for analysis—predisposed Bôcher to seek the guidance of that great leader.

On his return to Harvard as instructor he was entrusted with an elementary course. I use the word *entrusted* advisedly; for although Professor Byerly was willing that any one who was master of his subject should teach graduate stu-

* An address delivered before the Society at the request of the program committee, New York, December 28, 1928.

dents, he well knew that a beginner must be a born teacher if he is to teach beginners effectively. And Bôcher was a born teacher. On his mother's side he was of Puritan ancestry, and it is hard to imagine a Puritan mother who had not the call to teach. His father was professor of French, whose courses, because of the joy Professor Bôcher felt in the beauty of literature, made a wide appeal to the undergraduates of that day. Of the sternest scientific conscience, the younger Bôcher was keenly aware that, though truth may not be relative, the ability to recognize it is. With his broad and deep knowledge of his subject, whether he was teaching analytic geometry to freshmen or linear differential equations to graduates, he sought first of all to understand the student's domain of ideas, and then to give him the best which he could receive. It is generally recognized that interest in the science of mathematics and productive scholarship in that field redound to the good of elementary instruction. Bôcher proved the converse. Because he entered into the intellectual life of the freshman and devised ways and means to guide him in the pursuance of mathematics, he was thereby enabled to make his graduate instruction effective to an astonishing degree. When the *Annals of Mathematics* came to Cambridge, with Bôcher as the first editor, he sought and secured collections of problems in advanced subjects, results of such teaching as I have indicated in the higher fields. He encouraged, too, the preparation of expository articles.

He understood as few teachers do, the value and the difficulty of formulation. Think what a change would spread over the face of mathematical literature if writers had any adequate conception of formulation and the role it plays in clear thinking, and if they had the tact required to show them the relation of their message to the reader's knowledge. For most mathematicians are not intentionally mystics, either in the class-room or in the written word. They fail in the first requirement which science imposes on its followers, namely, to appreciate the importance of stating the problem definitely

and of seeing clearly the relation of the problem or theory to the rest of mathematics. I say, to *appreciate* the importance; for no improvement can come in presentation, whether it be a question of the author's own discoveries or of a college text-book, till writers *feel* the importance of clear formulation of results and of methods, and of clear insight into the inter-relations of the parts. It was this emotional element in Bôcher that at once made him a master in the class-room and directed his own energies into fruitful fields of research.

His *Algebra* was an outgrowth both of his teaching and of his own scientific work. When he gave his first advanced course in algebra (in 1898) there was no systematic treatment, in any language, of those parts of algebra which are most useful in their applications to geometry, to mathematical physics, and particularly to the calculus and analysis. The need of the student in this direction was clear to Bôcher, for he had himself, as a student, been obliged only recently to provide himself, unaided, with these tools. A paper on *Linear dependence and linear equations*, in the *Annals of Mathematics* for 1901, was his first published work in this field. It opens with the words:

The subject treated in the following pages is not only one which, owing to its numerous applications, is of considerable importance, but, while distinctly elementary, illustrates one of the most striking tendencies of modern algebraic and analytic work, namely, the tendency not to be satisfied with results which are merely true "in general," i.e., with more or less numerous exceptions, but to strive for theorems which are *always* true.

And his understanding of the needs of the reader is shown by the footnote which he appends to this section:

The great importance of this tendency will be apparent if we remember that when we apply a theorem, it is usually to a special case. If we know merely that the theorem is true "in general," we must first consider whether the special case to which we wish to apply it, is not one of the exceptional cases where the theorem fails.

Although his personal needs had not, so far as I am aware, led him to treat the factoring of polynomials in several variables, he recognized the importance of this subject, and his chapter on this topic gives precisely the information

which is needed for the study of the analogous problem of factoring analytic functions of several complex variables in the neighborhood of a point. Out of the systematic study of this latter problem has grown a theory of elimination which makes possible the proof of Weierstrass's second theorem on implicit functions in a single page.

The finish, both in form and substance, with which this book is written, can easily present a danger to the beginner, who has not yet learned that he must write his own book if he is to master the methods. But Bôcher's great skill in devising exercises, small and great, serve as a corrective for ill-founded self-confidence. As a sample of the kind of demand Bôcher encouraged his students to make on themselves, I may mention his paper in the *Annals of Mathematics* (October, 1905): *A problem in analytic geometry with a moral*.

Bôcher's first book, *Ueber die Reihenentwickelungen der Potentialtheorie*, appeared in 1894, and was a further development of the subject of his doctoral dissertation and prize essay. A perusal of the table of contents shows the fields with which he had become familiar: not projective geometry alone, but the geometry on a quadric manifold in higher spaces, with the method of elementary divisors for purposes of classification; the cyclides of Dupin and curvilinear coordinates, with extension to space of n dimensions; the partial differential equations of mathematical physics, leading him in his later work to consider that which is vital in integral equations and eventually to write a Cambridge Tract on this subject; total differential equations of the second order, linear and homogeneous, considered both in the complex domain and in reals, with reference to the theorems of oscillation of Sturm and Klein; Lamé's equation. It was in his later lectures on this part of the subject that one of his most gifted pupils obtained his start in mathematics. The development of arbitrary functions into series of standard types; systems of orthogonal functions, preparing the way for his later guidance of a student in the field of divergent

series and convergence factors, out of which studies have grown valuable contributions to this important subject.

It may seem that Bôcher imitated Klein, or that he was an adherent of a Göttingen School. But ideals cannot be imitated, and these ideals, in particular, are the spirit of the thought of free men, striving for a higher and deeper sense of scientific values in broad fields of mathematics.

Bôcher gave, at one time or another, nearly every one of the back-bone courses in the Department; but his special interest lay in the comprehensive course on modern geometry, to the development of which he contributed much, and on which, at the time of his death, he had begun to write a book. He gave, furthermore, for a number of years, one or the other of the introductory courses in the theory of functions of a complex variable, and of real variables. After Cole, he was the first at Harvard to give the former course from the geometric standpoint of Klein.

The latter course originated with him, being entitled *Infinite Series and Products*.* In the literature of that time every rigorous proof was given by ϵ 's, *ab ovo*. The fundamental theorems of analysis must, of course, be established by means of ϵ -proofs. But the derived theorems stand out more clearly when deduced from the former. It was not a question of logical elegance or of a doctrinal method, but rather of putting first things first, and helping the student to grasp the inherent difficulties of higher analysis. Bôcher had contempt for all that smacked of platform erudition, the cult peculiar to those who had a sense for rigor, of stating every theorem with the maximum of generality, at the expense of incisiveness with reference to its essential meaning.

* On looking up the date I find that I gave this course for the first time in 1897-98. The error is interesting, for it shows the close touch in which Bôcher and I were in organizing these courses. Bôcher had, for a year or two, brought into a half-course on the theory of equations and invariants a small amount of algebraic analysis, and he suggested the desirability of a half-course on infinite series and products—a suggestion in which I heartily concurred.

His advanced course was frequently on total linear differential equations of the second order, alternating between the complex theory and the real case. He also lectured repeatedly on the partial differential equations of mathematical physics, and on integral equations. Out of these courses grew his address at the Cambridge Congress (1912), and his Paris lectures (1913-14).

Throughout his whole academic career, beginning while he was still an instructor, he was actively engaged in guiding students in their work for the doctorate, and about twenty of these students completed their work under him. He was extraordinarily successful in discerning important topics, with range and depth, yet still within the powers of students at that stage of advancement. He was skillful in his guidance, not giving help that hampered the student's initiative or resourcefulness, nor neglecting him when he was ready to profit from a conference.

As editor of the *Transactions* he took a similar interest in the work of the younger mathematicians. His instinct for square dealing was shown anew by his impartial consideration of each paper submitted and his insistence on a prompt acceptance or rejection, followed in the former case by prompt publication. No referee could delay action long without hearing from Bôcher, and further delay caused the paper promptly to be recalled. He was no respecter of persons. Each paper was judged on its merits, and an opportunistic attitude toward contributors would have seemed to him sacrilege. He took great pains in helping unskillful writers to improve their presentation, but his first thought was for the genuineness of the scientific results obtained. Thus he contributed in those early days to make the traditions of the *Transactions* high in relation both to form and to substance, and to give the journal a reputation for fairness alike toward the older and the younger writers.

His work on the *Annals* had given him a valuable apprenticeship. It would have been easy to make the journal, when it came into the hands of the younger

mathematicians, interested as we all were in making for the Transactions the highest possible place in the mathematical world, a repository for papers not quite good enough for the latter journal. This was not Bôcher's view of the province of the Annals. A paper that had little to recommend it but the novelty of the result and the avoidance of logical errors in the proof would not pass muster. He set the same high standards of scientific merit which characterized the more advanced journal. The scope of the paper might be restricted, but not the genuineness of the results.

Nor were his sympathies and interests confined to his own courses and his special fields of study. At a time when mathematicians were just beginning to turn their attention to postulates, he recognized in the work of Huntington investigations of substantial character and used his influence to secure for Harvard the latter's services. Again, it was with full consciousness of the hearty moral support Bôcher accorded that Bouton built up his two courses on the Lie theory, and developed them to an extraordinarily high degree of effectiveness. He was well informed concerning the character of Coolidge's work, though he had no great technical knowledge of the subject, and I recall the keen interest he felt in it when Coolidge was a beginner. The same is true of his interest in the work of each of the others of his colleagues in the department,—Birkhoff, Jackson, Graustein,* and Gabriel Green. It is due in no small measure to these qualities in Bôcher, to his sense of justice and honesty of thought, to his willingness to accord others, irrespective of age or rank, what it seemed to him reasonable that they should accord him under like conditions, that our department has been able to pass from a patriarchal system to a pure democracy.

The student of mathematics would, indeed, be temerous, who would ignore the past and, like some modern novelists, attempt to build up a civilization out of his inner conscious-

* Professor Graustein was an instructor in the department for the year 1913-14.

ness. True, the seer has visions of what man may accomplish. But the visions which light the way to new achievement are those which spring from an understanding of what man has accomplished. They are not the dreams of a child, just getting its first glimpses of the world.

Bôcher sank his roots deep into the past of his science. He was a student of history, for he found in it an aid toward perspective, toward seeing mathematical truths in their relative meaning and their relative importance, and toward putting first things first. He shunned ill-defined speculations and held rather to the method of research in the natural sciences—that of defining precisely an important problem and bringing the knowledge of the day to bear on its solution. He may sometimes have been too little attracted by the desire to make a brilliant guess; but what he built is a part of the permanent structure of mathematics.

Creating for the mere sake of creating was a thing in which Bôcher did not believe. Productive scholarship he did believe in, and his whole life gave sustained evidence of this cardinal point in his scientific faith, this corner stone in his scientific character. But productive scholarship meant to him something larger and grander than the discovery of new truth. For new truths come to us in unassorted lots, and because a theorem is true, it does not follow that it has value. Productive scholarship meant to him a deeper insight into mathematics. And the deeper insight comes not merely from the new theorem, but chiefly from its contacts. Bôcher was keenly aware that new truth is under suspicion; that the great majority of new theorems which can not stand the test of usefulness, are unimportant, spurious, and will not survive the nurturing care of their discoverers. He strove rather to advance mathematics according to a well-defined program, refusing to be attracted by novelty for novelty's sake, when the science was in the greatest need of coordination of great truths already obtained and of deepening the understanding of important results as yet but imperfectly perceived.

For this reason Bôcher's work was not spectacular. It was basic. For example, in reporting on his scientific contributions, Professor Birkhoff remarks that none of Bôcher's published papers lay properly in the theory of functions of a complex variable. And yet, after the founders of analysis have been cited, no name appears more frequently in the footnotes of the speaker's treatise than that of Bôcher.

Perhaps nowhere in the domain of human thought is a cultural background more important than in mathematics. To train one's mind to recognize the great structures man has built, the great visions human thought has seen, in the realm of the greatest of the sciences, it is helpful to have first-hand knowledge of achievements in other fields of human thought and human action. As an undergraduate Bôcher informed himself of the then recent discoveries of Darwin—not as the philosopher or theologian conceives them, but as the scientist knows them—experiments with well defined boundary conditions; conclusions exhausting the results of the experiments; speculations suggesting further study, but not accepted as truth till rigorously established.

All through his life Bôcher turned to literature for recreation. He was fond of Shakespeare, of the German and the French classics, and of biography. Though not a performer, he had a good appreciation of music and derived much enjoyment from listening to the production of classical music. And what was true of his intellectual life outside the science was eminently true within it. His sympathies were broad and the amount of accurate information he possessed in various fields of mathematics extensive.

So it came about that Bôcher was able to render a great service to mathematics in America through his sense of values. Now, a sense of values is not something that is inherited, like the artistic temperament. It is an expression of character. Character is formed in the determined and sustained struggle for ideals. But if the ideals are not intelligently chosen; if one has not taken the pains to form

consciously a certain philosophy of life; to detach oneself from the present and inquire whence and whither; to estimate that which is great and to plan how one may strive intelligently for the highest which, under the given limitations of temperament and ability, one may hope to attain, the result must be mediocre. I never knew a time in Bôcher's life in which he was not working on the most important problem of those among which he could reasonably expect to make advance.

It was this habit of mind, these standards relating to scientific activity, accompanied as they were by his striving for the highest and the purest beauty in mathematics, that made him helpful to others. He wished to see the influence of the *Transactions*, particularly on the younger writers, directed into these channels, and in this I am well aware that he had the earnest cooperation of our colleague and President, Van Vleck. What he lacked was the zeal of the reformer, who believes that men can be made to improve their ideals by having the truth thrust at them. He gave freely of himself to all who, in the spirit of truth, sought to discuss the truth with him. He felt no inner urge to make men do what of their own initiative they could not. Among the strongest of his contemporaries, however, his influence was strong, and he found hearty support from some of the best men of his time. His pupils have rendered a good account of themselves, and he is remembered by them as an inspiring teacher, not through emotional appeal, but through the genuineness of his mathematical perceptions and ideals, his sincerity, the rugged honesty of his thought, and his unswerving devotion to his science.

I may seem to have laid too much stress on Bôcher's appreciation of the work of others and his efforts to clarify and unify it, and the impression can easily be created that he absorbed knowledge rather than that he smote from the rock the living waters, that he had imagination of high order, and that he produced original results of high merit. It is not easy to correct this impression by citing specific results readily in-

telligible to the lay mathematician. All of Bôcher's more important work was based on theories which had already attained a high degree of advancement. To understand his precise contribution, it is often necessary to make a comprehensive study of a whole field, or, indeed, of different fields, of mathematics. But then it will be found that he brought an originality of thought and method which secure for him a high place among the mathematicians of his time. His scientific biographer, Professor Birkhoff, closes his estimate with the words:

"In amount and quality, his production exceeds that of any American mathematician of earlier date in the field of pure mathematics."

The contemplation of Bôcher's life leads us to the question: Is mathematics a collection of concepts and relations between these concepts—a *logical system*, in the broadest interpretation of this notion, as formulated by Bôcher himself in his St. Louis address—or is not a truer picture that of the various civilizations through which the human race has passed and is continuing to pass? These civilizations develop gradually, often at great expense, a purer conception of beauty, a deeper insight into truth, a perception of the eternal verities, a recognition of the absolute invariants of the human mind and of the human soul. And when these invariants have once been discerned, they pass over into the traditions of that people. They become a part of the intellectual and the spiritual heritage of the race.

Bôcher valued a new result, but he did not find the satisfaction in a collection of new results, or even new theories, which the child finds in a stamp collection. He looked beyond the new result to its place in the science, and his powers of criticism were such that he saw with remarkable clearness the bearing of that which is new on the development of the science.

Critical is a treacherous word to apply to a scholar, for it suggests destructive criticism and the cavilling of literary men. No greater mistake could be made than to think of

Bôcher as belonging to that category of critics. Constructive in all his thinking, he made his criticism of the work of others productive of new achievement on their part. He knew, too, how to profit from criticism of his work by others, even though given in an unfriendly spirit. It was the content, not the form, to which he chose to direct his attention.

We have witnessed the revolt of youth in society, the tearing of the mask from the conventions of the past, in these days since the War, and some of the older ones of us have felt humbled at the thought of our failure to make clear to those who come after us the eternal verities which are the heritage of the race. But each new generation must find these truths, and if it will not learn from the past, then it must create again the experience on which the wisdom of the past was founded. For build man must, whether for the hour or for eternity.

No less in mathematics have we a rich heritage from the past, and the desire of youth to go forward. The great problem is to stir the soul of youth to understand the former, while retaining in full measure the latter. That which Bôcher has left us has permanent value in this great field of human endeavor.

Bôcher's influence on American mathematics will wax with time, if we believe in progress. His influence is needed today. Not in economics and in ethics alone are the results of the Great War manifest. There is in mathematics an absence of scientific tranquility, a desire to get results without paying the price of scientific productivity. It is with satisfaction and inspiration that one turns to Bôcher's writings and Bôcher's teachings. The younger generation of mathematicians is not lacking in those who understand the message, and to them we may confidently entrust the future of mathematics in America.

We look to Bôcher as an example of one whose highest satisfaction was in service, but whose conception of service was derived from the highest needs of the science and of its disciples. To understand before undertaking; to examine

objectively the products of imagination; to make that which is genuine accessible for those who come after—these are the three great traits of his intellectual being.

A tribute to Bôcher would be imperfect which did not lay emphasis, not only on his high idealism and his intellectual achievements, but also on the character of the man, his passion for truth, his freedom from self-seeking, his high conception of duty and his loyalty to his fellowmen, above all, his faithfulness in little things. To him the words apply:

How happy is he born or taught
Who serveth not another's will;
Whose armor is his honest thought,
And simple truth his highest skill;

Whose passions not his masters are;
Whose soul is still prepared for death,
Untied unto the world by care
Of prince's ear or vulgar breath;

Who hath his life from rumors freed,
Whose conscience is his strong retreat,
Whose state can neither flatterers feed,
Nor ruin make oppressors great;

Who God doth late and early pray
More of his grace than goods to lend;
And walks with man, from day to day,
As with a brother and a friend.

This man is freed from servile bands
Of hope to rise, or fear to fall;
Lord of himself, though not of lands,
And having nothing, yet hath all.

HARVARD UNIVERSITY