NON-EUCLIDEAN GEOMETRY, A RETROSPECT*

BY JAMES PIERPONT

1. *Introduction.* Non-euclidean geometry had its origin in the unwearied attempts of mathematicians during 2000 years to free the *Elements* from its one notorious blemish, namely, the postulate or axiom relating to parallel lines.

Wallis (1693), Saccheri (1733), Lambert (1786), Legendre (1794), Schweikert (1807), Wachter (1817), and Taurinus (1826) are noteworthy forerunners, but the first systematic and rigorous development of the subject to be published was a series of five papers entitled *On the principles of geometry*, which appeared during 1829–30 in the Kasan Messenger. Their author was an unknown Russian mathematician, Lobachevsky. The present year 1929 may be regarded as rounding out the first century of this new science. What has it accomplished in this time? I propose to answer this question, paying attention not so much to concrete results obtained, as to the basal ideas which have made its phenomenal progress possible.

The early history of our subject is too well known to require more than a few words. We must mention, however, that Gauss was already in full possession of Lobachevsky's results, although he permitted nothing to reach the ears of the public. The Hungarian Bolyai had also broken through the barriers of euclidean geometry independently. His treatise, through no fault of his own, did not appear till 1832.

The method used by these geometers was the synthetic method of Euclid, with a slight mixture of trigonometry, analytic geometry and the calculus. It was adequate to establish the existence of a non-euclidean geometry, but new methods and ideas were necessary for further progress. These came with a bound.

Riemann's Habilitationsschrift *Ueber die Hypothesen welche der Geometrie zu Grunde liegen*, read before the Philosophical Faculty of the University of Goettingen in 1854, but first

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published in 1868, after Riemann's death; Cayley's *Sixth memoir upon quantics* (1859); Klein's *Ueber die sogenannte Nicht-Euklidische Geometrie* (1871): these were the epoch making papers of this second period, which we now consider.

2. *The Projective Methods of Cayley and Klein.* Cayley takes four numbers x_1, x_2, x_3, x_4, and regards their ratios $x_1:x_2:x_3:x_4$ as defining a point. In the notes to volume 2 which Cayley prepared for his *Collected Papers*, he says on page 605:

"As to my memoir, the point of view was that I regarded 'coordinates' not as distances or ratios of distances, but as an assumed fundamental notion not requiring or admitting of explanation."

This abstract method of procedure seems to have been entirely misunderstood by Klein and as a result he has developed non-euclidean geometry from another point of view, as we shall see.

Continuing with Cayley, we define a straight abstractly by the points

$$x_i = la_i + mb_i, \qquad (i = 1,2,3,4).$$

Here l, m are parameters and a_i, b_i the coordinates of two points a, b. A plane is defined by the equation

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0.$$

To define distance and angle, Cayley introduces the quadratic form

$$F(x,x) = (x,x) = \sum a_{ij}x_ix_j, \quad a_{ij} = a_{ji}, \quad (i,j = 1,2,3,4),$$

and the bilinear form

$$F(x,y) = (x,y) = \sum a_{ij}x_iy_j.$$

In euclidean geometry, $F(x, x) = 0$ defines a quadric surface which Cayley calls the *absolute*.

Associated with the form F is the form

$$G(u,u) = \sum a^{ij}u_iu_j,$$

in which a^{ij} = minor a_{ij}/a, where a denotes the determinant $|a_{ij}|$. Then $G=0$ is the equation of the absolute in plane coordinates. Cayley now defines the distance between two points x, y by means of the formula

$$(1) \qquad \cos \delta = \frac{F(xy)}{(F(xx)F(yy))^{1/2}},$$

and the angle between two planes u, v by the formula

$$(2) \qquad \cos \phi = \frac{G(u,v)}{(G(uu)G(vv))^{1/2}} \cdot$$

With these definitions and those which naturally flow from them, it is possible to construct a complete and simple development of non-euclidean geometry.*

We turn now to Klein. He tells us in his notes to his papers on non-euclidean geometry (Mathematische Abhandlungen, vol. 1, pp. 50–52, 241–243) that he was introduced to this subject while attending Kummer's and Weierstrass' seminar in Berlin (1869–70) by O. Stolz, who was thoroughly familiar with the work of Lobachevsky, Bolyai, and von Staudt.

Klein soon saw the relation between the geometry of Lobachevsky and Bolyai and Cayley's memoir of 1859; and in 1871 he published his first paper on this subject, as already noted. His Goettingen lectures on *Nicht-Euklidische Geometrie* were published in autograph form in 1892. These have enjoyed the most widespread popularity, and probably the majority of mathematicians of the younger generation learned their non-euclidean geometry from these lectures.

Klein apparently was influenced by a youthful paper of Laguerre (1853) which contains implicitly this result: The angle ϕ made by two straights a, b meeting at the point x is expressed by $(i/2) \log (ab, u'u'')$, where u', u'' are the straights joining x with the cyclic points, and $(ab, u'u'')$ is the cross ratio of the four straights.

The equation of u', u'' is $u_1^2 + u_2^2 = 0$, a degenerate conic. Following Cayley, Klein replaces this conic by the general conic $F = \sum a_{ij} x_i x_j = 0$ in point coordinates, or $G = \sum a^{ij} u_i u_j = 0$ in line coordinates; and he defines the distance δ between two points x, y in the plane by the formula

$$(3) \qquad \delta = \frac{c}{i} \log (xy, x'x'').$$

* See the forthcoming paper by the author, *Cayley's definition of non-euclidean geometry.*

Here x', x'' are the points of intersection of the straight lx_i+my_i with the conic $F=0$, $(xy,\ x'x'')$ is the cross ratio of the points in the parenthesis, and c is a constant chosen so that δ may be real. Similarly the angle ϕ between two straights u, v, meeting at point x, is defined by the equation

$$\phi = \frac{c'}{i} \log\,(uv, u'u''). \tag{4}$$

Here u', u'' are the tangents drawn from x to the conic F or G, $(uv,\ u'u'')$ is the cross ratio of the straights in the parenthesis, and c' is a constant chosen so that ϕ may be real.

If we take $F=0$ to be $x_1^2+x_2^2+x_3^2=0$, we get the geometry of Riemann, which we shall discuss later; if we take $F=0$ to be $x_1^2+x_2^2-x_3^2=0$, we get the geometry of Lobachevsky and Bolyai. The extension of the foregoing to space is obvious. The expressions (3), (4) may be immediately transformed into those of (1), (2).

It appears at first sight as if little had been gained. Klein's real contribution is the following. The cross ratio $(xy,\ x'x'')$ in (3) depends on the coordinates of the four points x, y, x', x'', and also on the lengths of their segments. The expression (3) therefore defines a non-euclidean geometry by means of euclidean geometry. This would be a serious defect if it could not be obviated. This Klein has done employing the ideas of von Staudt, freed from the parallel axiom. If one wishes to carry this work through rigorously, it requires a very considerable effort which most mathematicians, I fancy, would be glad to avoid, as in fact I believe they do. It seems strange therefore that the simpler method of Cayley has been entirely overlooked. It should be noted that the superiority of the methods of Cayley and Klein over the synthetic methods of the founders of non-euclidean geometry is exactly analogous to the superiority of projective methods over the synthetic methods of Euclid in euclidean geometry. Another enormous advantage is due to the fact that many of the methods and results of euclidean projective geometry may be taken over in toto, or with obvious modifications, without putting pen to paper.

3. *The Differential Method. First Period. Riemannian Geometry.* This period begins with the publication (1868) of

Riemann's Habilitationsschrift mentioned above. Riemann's paper may be regarded as a vast generalization of Gauss' *Disquisitiones generales circa superficies curvas* (1827). It is interesting to note that Gauss was present when Riemann read his paper. Riemann's memoir, when finally published, was promptly taken up. Beltrami (1868), Christoffel and Lipschitz (1869), Schlaefli (1871), Beez (1874), Voss (1880), Ricci (1884), and Killing (1885) were among the first to enter this field. In 1901 Ricci and Levi-Civita published in volume 54 of the Mathematische Annalen a resumé of their new tensor analysis.

Riemann's point of departure is analogous to that of Cayley. Any n numbers $x_1, x_2, \cdots, x_n$ he regards as the coordinates of a point. When the x's vary in a continuous manner, the corresponding points range over a part or all of an n-way space. Riemann offers no geometric picture of his configurations; they are abstractions determined by arithmetic relations between the coordinates, *which latter are undefined.*

Riemann regards this highly abstract procedure as quite necessary, for at the close of his paper (Werke, p. 268, edition of 1876) he says: "Solche Untersuchungen, welche, wie hier ausgeführt, von allgemeinen Begriffen ausgehen, können nur dazu dienen, dass diese Arbeit nicht durch die Beschränktheit der Begriffe gehindert und der Fortschritt im Erkennen des Zusammenhangs der Dinge nicht durch überlieferte Vorurtheile gehemmt wird."

We cannot go into details, as that would take too much space,* but a few words may be spent on a fundamental question. Riemann regards the metric properties of his space to be defined by the distance between two nearby points $x, x+dx$. He takes this to be defined by the formula

$$(5) \qquad ds^2 = \sum a_{ij} dx_i dx_j, \qquad a_{ij} = a_{ji}, \qquad (i,j = 1,2,\cdots,n),$$

where the a's are functions of $x_1, \cdots, x_n$ whose determinant a is different from zero. For euclidean space of three dimensions, we would have

$$(6) \qquad ds^2 = dx_1^2 + dx_2^2 + dx_3^2 .$$

* The reader may if he chooses consult the author's paper, *Some modern views of space*, this Bulletin, vol. 32 (1926), pp. 225–258.

A curve in Riemann space is defined by n relations of the type $x_i = x_i(t)$; a straight is a curve along which $\delta\int ds = 0$, where ds is defined by (5).

When the variation is executed we are led to n differential equations defining the straight, namely,

$$(7) \qquad \frac{d^2 x_i}{ds^2} + \sum_{\lambda\mu} \begin{Bmatrix} \lambda\mu \\ i \end{Bmatrix} \frac{dx_\lambda}{ds}\frac{dx_\mu}{ds} = 0, \qquad (i, \lambda, \mu = 1, 2, \cdots, n).$$

The symbols under the summation sign were introduced by Christoffel, who was one of the earliest to elaborate the ideas of Riemann. He sets

$$(8) \qquad \begin{bmatrix} \alpha\beta \\ i \end{bmatrix} = \frac{1}{2}\left(\frac{\partial a_{\alpha i}}{\partial x_\beta} + \frac{\partial a_{\beta i}}{\partial x_\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x_i}\right).$$

If we denote by $a^{\lambda\mu}$ the minor of $a_{\lambda\mu}$ in a, divided by a, we have

$$(9) \qquad \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix} = \sum_i a^{i\lambda} \begin{bmatrix} \alpha\beta \\ i \end{bmatrix}.$$

In the following we shall set

$$(10) \qquad \begin{bmatrix} \alpha\beta \\ i \end{bmatrix} = C_{\alpha\beta,i}, \qquad \begin{Bmatrix} \alpha\beta \\ i \end{Bmatrix} = C^{i}_{\alpha\beta}.$$

The quantities $\xi_\lambda = dx_\lambda/ds$ we call the direction parameters of the straight. Let ξ_λ, η_λ be the parameters of two straights meeting at a point. The straights $l\xi_\lambda + m\eta_\lambda$, where l and m are parameters, define a plane. Other geometrical terms are introduced in a similar manner, as generalizations of the corresponding notions in euclidean geometry.

The question at once arises: when do two quadratic differential forms of the type (5) define the same geometry? It is well known that the geometry on a cylinder or a cone, or more generally on a developable surface, is, for not too large regions, identical with that of the plane, that is, it is euclidean. More generally, we may ask: when is the geometry on two given surfaces the same (for not too extended regions)? This question was partly answered by Gauss by means of the important notion of total curvature or Gaussian curvature k, at a point on a surface. If the geometries of two surfaces have constant curvature, this condition is sufficient. Gauss established the important fact that k is an invariant of the quadratic form

$$c_{11}du_1^2 + 2c_{12}du_1du_2 + c_{22}du_2^2, \tag{11}$$

defining the metric of the surface. The determinant of this form we will call c. We can write down the value of k in a neat form by introducing what is now known as the Riemannian curvature tensor. For the general quadratic form (5), this tensor has n components (not all distinct) defined by the equations

$$R_{\mu\lambda,jk} = \frac{\partial C_{\lambda j,\mu}}{\partial x_k} - \frac{\partial C_{\lambda k,\mu}}{\partial x_j} + \sum_{i\alpha}(C_{\lambda k,\alpha}C_{\mu j,i} - C_{\lambda j,\alpha}C_{\mu k,i})a^{i\alpha}, \tag{12}$$

where the C's are the Christoffel symbols (10) relative to the form (5), and $i, j, \alpha, \lambda, \mu = 1, 2, \cdots, n$. If we denote by $R_{\mu\lambda,jk}(c)$ the components of the Riemannian tensor relative to the binary form (11), the Gaussian curvature of a surface in euclidean geometry is

$$k = R_{21,12}(c)/c. \tag{13}$$

Guided by the ideas of Gauss, as developed in the *Disquisitiones generales*, Riemann proceeds as follows. Any two geodesics ξ, η in his general n-space (5) meeting at a point P define a plane $\tilde{\omega} = l\xi + m\eta$. This being a two-way manifold, the coordinates $x_1, x_2, \cdots, x_n$ of any point on it are functions of two parameters, say u_1, u_2. Hence

$$dx_i = \frac{\partial x_i}{\partial u_1}du_1 + \frac{\partial x_i}{\partial u_2}du_2, \quad (i = 1, 2, \cdots, n).$$

These set in (5) show that the metric of the plane $\tilde{\omega}$ is defined by a form of the same type as (11), say by

$$ds^2 = b_{11}du_1^2 + 2b_{12}du_1du_2 + b_{22}du_2^2, \tag{14}$$

whose determinant call b. The coefficients of this form depend on u_1, u_2, and the $2n$ direction parameters $\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n$.

Riemann defines the curvature of $\tilde{\omega}$ to be

$$k = R(b)_{21,12}/b, \tag{15}$$

where the $R(b)$ here refers to the form (14). We see that this is entirely analogous to Gauss' definition (13). Here, however, the surface $\tilde{\omega}$ lies in an n-way space whose metric is given by (5). Suppose we turn the plane $\tilde{\omega}$ about P by replacing the straights ξ, η by other straights meeting at P. Each of these planes will have a curvature k. If all these k's are equal at P, we say that our space (5) has curvature k at this point. If k has the same value at all points P of our space, we say the space (5) is of constant curvature k.

For such spaces, we have the result Gauss established for $n=2$, that is, two n-way spaces of the same constant curvature have the same geometry (for not too large regions). Spaces of constant curvature have also this important property: in them, rigid bodies exist which may be moved about freely without any distortion. This of course is not generally true. For example, on an oval surface of revolution, like an egg, a small figure drawn on it can be moved without distortion only by revolving it about the axes of the surface. Riemann found that by a proper choice of variables the metric (5) of a space of constant curvature could be given the form

$$(16) \qquad ds^2 = \frac{dx_1^2 + dx_2^2 + \cdots + dx_n^2}{[1 + ek(x_1^2 + x_2^2 + \cdots + x_n^2)]^2}.$$

For $k=0$, and $n=3$, this reduces to (6); thus euclidean space is a space of zero curvature. If in (16) we take $e=-1$, $n=3$, we get the space of Lobachevsky and Bolyai. As the pseudosphere has a constant negative Gaussian curvature, the geometry on this surface is identical (for not too large regions) with plane Lobachevskian geometry, as first remarked by Beltrami (1868).

If we take $e=+1$, $n=2$, we have the geometry on a sphere. For $e=+1$, and $n=3$ we have a three-dimensional geometry never before imagined. As on the sphere, so here in three-way space, all straights are closed curves of constant length $\pi/\sqrt{k}$. Space is finite in extent, but without boundaries; its volume in fact is $\pi^2/(4k^{3/2})$.

This new space with positive k may be called spherical. As Klein first remarked, it has two forms; in the second form the straights are still closed, but of length $\pi/(2\sqrt{k})$, two points always determine uniquely a straight; whereas in the first form there is an exception, namely, when the two points are at a distance $\pi/(2\sqrt{k})$. The connectivity of this second form of spherical space is quite complicated. For example, a watch moved along a straight l has been rotated through 180° about l when it returns to its point of departure. We have dwelt on spaces of constant curvature, partly because of their relation to the results obtained by projective methods, and partly because of their intrinsic value. But the geometry of spaces of constant curvature is only a small part of Riemannian geometry, as it is now called; that is, the geometry based on the general metric (5). In this geometry, the notions of euclidean space are extended so as to apply to this general metric. The reader who wishes to study this subject may consult one of the treatises mentioned in the footnote.*

4. *The Differential Method. Second Period. Non-Riemannian Geometry.* Riemannian geometry is based on a quadratic differential form. For a long time it was studied as an abstract science, just as the ancient Greeks studied the conic sections, or Gauss and Kummer studied the theory of numbers. Then all at once a great change came. In 1916 Einstein promulgated his general theory of relativity. The basis of this theory is a quadratic differential form (5) in four variables. This gave Riemannian geometry an enormous impetus. Riemannian geometry was no longer an abstract theory cultivated by a small body of mathematicians and quite neglected by the others; of a sudden it became of great practical value and widespread interest.

In 1917, Levi-Civita introduced the notion of infinitesimal parallel displacement. It was soon seen that this notion is more fundamental than that of the notion of distance ds between two nearby points, and that it can be used to find a geometry more general than that heretofore considered, a geometry now called non-Riemannian. It contains Riemannian geometry as a special case.

It is interesting to discover the origin of a notion so fundamental. Thompson and Tait in their *Natural Philosophy* (edition of 1879) discuss the kinematics of a surface rolling on a

* J. L. Coolidge, *The Elements of Non-Euclidean Geometry*, especially Chapters XV, XVI, Clarendon Press, 1909. L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1926. W. Killing, *Die Nicht-Euklidischen Raumformen*, Leipzig, 1885.

plane without spin. They say in § 135: "The whole change of direction in a curved surface from one end to another of any arc of a curve traced on it is equal to the change of direction from end to end of the trace of this arc on a plane by pure rolling." In 1906 Brouwer hit upon the same notion for spaces of constant curvature. We may define the new notion as follows.

Suppose an elementary vector ξ whose components are $\xi_1, \xi_2, \cdots, \xi_n$ is moved along a curve K from a point x to the point $x+dx$, a distance ds as given by (5). If this vector changes its direction so that its components satisfy the n relations

$$\frac{d\xi_i}{ds} + \sum_{jl} C^{i}_{jl}\xi_l \frac{dx_j}{ds} = 0, \quad (i,j,k = 1,2,\cdots,n), \tag{17}$$

the C's being the Christoffel symbols (10) belonging to the metric (5), we say that ξ has received an infinitesimal parallel displacement along K. It is also said to be displaced geodetically along K. In fact the direction parameters $\xi_i = dx_i/ds$ of the tangent to a geodesic satisfy the equations (17) by virtue of (7). Hence the tangent to a geodesic receives an infinitesimal parallel displacement as it moves from x to $x+dx$ along the curve.

When ξ is moved geodetically from A to B along two different curves, we find the direction of ξ at B is not the same. If ξ is displaced geodetically along a closed curve G its change in direction after the circuit is given by the equation

$$\Delta\xi_i = \int_G \sum_{jk\lambda} \xi_\lambda R^{i}_{\lambda jk} dx_j dx_k, \tag{18}$$

where

$$R^{i}_{\lambda jk} = \frac{\partial C^{i}_{\lambda j}}{\partial x_k} - \frac{\partial C^{i}_{\lambda k}}{\partial x_j} + \sum_{\alpha}(C^{i}_{\alpha k}C^{\alpha}_{\lambda j} - C^{i}_{\alpha j}C^{\alpha}_{\lambda k}). \tag{19}$$

The relations (17), (18), (19) suppose that a metric (5) has been given us in advance. We may, however, proceed in another order. Let us suppose instead of starting with the quadratic form (5) and forming then the C^{i}_{ij} and the $R^{i}_{\lambda jk}$, we take n^3 functions of $x_1, x_2, \cdots, x_n$ which we will denote by Γ^{i}_{jl}, and say that the vector ξ is displaced geodetically from a point x to $x+dx$ on a curve H when

$$d\xi_i + \sum_{jl} \Gamma^{i}_{jl}\xi_l dx_j = 0. \tag{20}$$

Let us define a geodesic to be such a curve that its tangent is displaced according to (20). When a vector ξ is displaced according to (20), that is, geodetically, around a closed circuit G, we find that its components receive the increments

$$\Delta\xi = \int_G \sum_{jkl} \xi_\lambda K^{i}_{\lambda jk} dx_j dx_k,$$

where the K's are obtained from (19) by replacing the C's by the new functions Γ^i_{jl}. With these Γ's and K's it is possible to develop a geometry in many ways analogous to that of Riemann. Still further generalizations have been made but these few remarks must suffice.*

In closing, the author expresses the hope that the present sketch may awaken a desire in the reader to acquaint himself more fully with this fascinating subject. At least it will give him an idea what tremendous strides non-euclidean geometry has made since Lobachevsky published his *Principles of Geometry* in 1829.

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* For further information the reader may consult the author's paper referred to in an earlier footnote, or L. P. Eisenhart, *Non-Riemannian Geometry*, Princeton University Press, 1928; D. J. Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie*, Berlin, 1922.