the formulas (11), (8), and (5) furnish us a solution of the problem.

We still have to consider the possibility of satisfying the equations (4) by making vanish more than one of the expressions in brackets in the left hand sides of these equations and less than two of the numbers a,b,c,d; but it is easy to see that this is possible only in the exceptional case mentioned above, namely, where two of the numbers f,g,h,k are equal. Barring this exceptional case we have then a unique system of values for e, a^2, b^2, c^2, and d^2; but we can choose the signs of c and d.

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ON THE PHRAGMÉN-BROUWER THEOREM*

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1. *Introduction.* The purpose of this note is to give an elementary demonstration of the validity of the Phragmén-Brouwer theorem in spaces satisfying certain conditions. This theorem has been proved and generalized by Urysohn and Alexandroff† for cartesian spaces by means of the theory of dimensionality, but the great importance of the theorem seems to the writer to justify the offering of another proof which involves only the elementary principles of the point-aggregate theory. As is well known, to prove this theorem for a space is the same as demonstrating that, if m and n are any two points of the space and C is an irreducible cut between m and n, then C is a continuum. It is this form which will be used and the proof is deduced from the validity of the theorem in the euclidean plane.

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† P. Alexandroff, *Sur les multiplicités cantoriennes et le théorème de Phragmén-Brouwer généralisé*, Comptes Rendus, vol. 183, pp. 722–724.

2. *Notation.* In addition to the familiar notations of the point-aggregate theory, the following will be used.

If a and b are two points, the distance between them is denoted by Dist (a,b).

The symbol $V_\epsilon(a)$ denotes the set of points whose distances from a are less than ϵ.

If C is a closed cut of space between the points (or point sets) A and B, we write that C is an $S(A,B)$.

The symbol $\overline{\lim} \, A_i$ denotes the upper closed limiting set of the sequence of sets $\{A_i\}$ as $i \to \infty$.

3. *Convex Spaces.* Let a and b be points of a metric space Z and let $k = $ Dist (a,b). For each value of t, $0 \leq t \leq k$, let there be one and only one point x such that Dist $(a,x) = t$ and Dist $(x,b) = k - t$. If this property pertains to each pair of points of Z, we say that Z is *convex.** The set of points $\{x\}$ is called a *linear segment*, or simply a *segment*, and is denoted by ab. Thus a convex space is one in which each pair of points determines a unique segment.

Every euclidean space is convex, but not conversely. As an example, consider a space consisting of three ordinary linear segments ax, bx, and cx lying in a euclidean plane. Let the distance between a pair of points on any one of these be measured as usual. If u, v, w are points on ax, bx, and cx, respectively, let Dist (u,v) be defined as Dist $(u,x) +$ Dist (v,x), etc. In this space any two points can be joined by a unique segment, but we notice that two different segments, as uv and uw, may have a segment, here ux, in common.

The following properties are readily proved for any convex space in which every bounded set has at least one limiting point:

(a) it is a continuum;

(b) it is locally connected at each point;

(c) if $\{x_i\}$ and $\{y_i\}$ are sequences of points converging to x and y, respectively, Dist $(x_i, y_i) \to$ Dist (x, y) and $x_i y_i \to xy$, that is, the set xy is both the upper and the lower closed limiting set of the sequence of sets $\{x_i y_i\}$.

A space of this kind obviously includes euclidean space of any number of dimensions as a special case.

* This term is used in a slightly more restricted sense by Hausdorff; see *Mengenlehre*, p. 96.

4. THEOREM. *Let Z be a convex space in which every bounded infinite set has at least one limiting point. If m and n are two points and C is an irreducible $S(m,n)$, then C is a continuum.*

PROOF. We assume the opposite and arrive at a contradiction. Let $C = A + B$, where A and B are closed and $A \cdot B = 0$. Let M and N be the components of $Z - C$ containing m and n, respectively. Since Z is everywhere locally connected, M and N are connected regions and C is the frontier of each of them.

Let h be a point of A and k one of B, and let $\epsilon > 0$ be less than one third of the smaller of the quantities Dist (h,B) and Dist (k,A). Then $B \cdot V_\epsilon(h) = 0 = A \cdot V_\epsilon(k)$. Now there is a point a in M and a point c in N such that the segment $ac \subset V_\epsilon(h)$. Likewise there is a point b in M and a point d in N such that the segment $bd \subset V_\epsilon(k)$. Then $ac \cdot B$, $bd \cdot A$, and $ac \cdot bd$ are void. As C is an $S(m,n)$, we have $ac \cdot A \neq 0$ and $bd \cdot B \neq 0$. Since Z is everywhere locally connected, there are simple arcs ab and cd in M and N, respectively. Finally, there is no loss in generality in assuming that $(ab+cd) \cdot (ac+db) = a+b+c+d$. For $ac \cdot ab$ and $ac \cdot cd$ are closed sets; hence ac contains a segment $a'c'$ such that $a'c' \cdot ab = a'$ and $a'c' \cdot cd = c'$. Likewise, bd contains a segment $b'd'$ such that $b'd' \cdot ab = b'$ and $b'd' \cdot cd = d'$. Let $a'b'$ be the sub-arc of ab whose ends are a' and b', and $c'd'$ have a similar relation to cd. Clearly $a'b'$, $a'c'$, $b'd'$, and $c'd'$ have the properties assigned to ab, ac, bd, and cd.

In a cartesian plane let S be the closed square whose frontier is the union of the segments pq, pr, qs, and rs, each of unit length. Set up a homeomorphism between ab and pq, and between cd and rs, so that a, b, c, and d correspond to p, q, r, and s, respectively. Now let u be any point of pq and v one of rs so that uv is parallel to pr. In Z let x and y be the points of ab and cd corresponding to u and v, respectively, and let xy be the segment joining them. (If $u = p$, $xy = ac$; if $u = q$, $xy = bd$.) Let Q be the union of the sets $\{xy\}$ for all positions of u on pq. Set up a homeomorphism between each xy and the corresponding uv such that, if z is a point of xy and t is the corresponding point of uv, Dist $(u,t) = $ Dist $(x,z)/$Dist (x,y). Then to each point of S there corresponds exactly one point of Q. In general the converse is not true, for two or more different segments $\{xy\}$ may have common points.

We proceed to prove that Q is the continuous image of S. Let $t_i \rightarrow t$ in S and let z_i and z be the corresponding points in Q; we must prove that $z_i \rightarrow z$. Let t_i lie on $u_i v_i$, t lie on uv, $x_i y_i$ correspond to $u_i v_i$, and xy correspond to uv. It is obvious that $u_i \rightarrow u$, $v_i \rightarrow v$, whence $x_i \rightarrow x$, $y_i \rightarrow y$, and $x_i y_i \rightarrow xy$. Hence, whether or not the sequence $\{z_i\}$ converges, $\overline{\lim} z_i \subset xy$.

By construction Dist $(x_i,\ z_i) = $ Dist $(x_i,\ y_i) \times$ Dist $(u_i,\ t_i)$. By the previous paragraph Dist $(x_i,\ y_i) \rightarrow$ Dist $(x,\ y)$ and also Dist $(u_i,\ t_i) \rightarrow$ Dist $(u,\ t)$. Since Q is bounded, the sequence $\{z_i\}$ has a convergent sub-sequence $\{z_i'\}$ whose limit we call z'. The results just obtained show that Dist $(x,\ z') = $ Dist $(x,\ y)$ $\times$ Dist $(u,\ t)$, or Dist $(u,\ t) = $ Dist $(x,\ z')/$Dist $(x,\ y)$. The definition of the correspondence between Q and S gives immediately that $z' = z$. As this holds for every convergent sub-sequence of $\{z_i\}$, it follows that $z_i \rightarrow z$ and hence that the correspondence is continuous.

Let $A' = A \cdot Q$ and $B' = B \cdot Q$. Then $A' \cdot bd = 0$, $B' \cdot ac = 0$, and $(A' + B') \cdot (ab + cd) = 0$. Let E be the set of all the points of S whose images in Q are points of A', and let F have a like relation to B'. Then $E \cdot pr \neq 0$, $F \cdot qs \neq 0$, $E \cdot qs = 0$, $F \cdot pr = 0$ and $(E + F) \cdot (pq + rs) = 0$. Since the correspondence is continuous, E and F are closed; since $A' \cdot B' = 0$, $E \cdot F = 0$.

As $E + F$ does not contain a continuum joining pr to qs, it follows from the properties of the euclidean plane that there is a continuum K in $S - (E + F)$ such that $K \cdot rs \neq 0$ and $K \cdot pq \neq 0$. Let L be the image of K in Q. Then $L \cdot (A' + B') = 0$ and L is a continuum. As $L \subset Q$, $L \cdot (A + B) = 0$. Since $K \cdot rs \neq 0$ and since $K \cdot pq \neq 0$, $L \cdot ab \neq 0$ and $L \cdot cd \neq 0$.

But $ab \subset M$ and $cd \subset N$, whence it follows that $C = A + B$ is not an $S(M, N)$, contrary to hypothesis. Thus the theorem is proved.

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