THE RATIONALITY OF CERTAIN CONTINUOUS CURVES

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1. *Introduction.* In this paper we establish the following theorem.

THEOREM. *Every plane continuous curve M every subcontinuum of which is a continuous curve is a rational curve.*†

We first state two lemmas, both of which are very easy to prove.

LEMMA 1. *In order that any continuous curve M should be a rational curve it is necessary and sufficient that every maximal cyclic curve of M be a rational curve.*

LEMMA 2. *If P is any point of a plane continuous curve M , every subcontinuum of which is a continuous curve, then there exists a continuous curve M^* , also every subcontinuum of which is a continuous curve, which contains M and such that P is on the boundary of no complementary domain of M^* .*

The necessity of the condition in Lemma 1 is obvious. The sufficiency is established by the following steps: (a) a continuum M is a rational curve if and only if every two points A and B of M can be separated in M by a countable subset of M . (Menger, loc. cit., proves this where A and B are closed sets; a simple application of the Lindelöf Theorem suffices to prove the part not included in Menger's theorem); (b) if two points A and B of a continuous curve M lying together in a maximal cyclic curve C of M are separated in C by a closed subset K of C , then K also separates A and B in M , (because if not, then A and B lie together in a component H of $M - K$, and since $H \cdot C$

† A continuum M is said to be a rational curve if each point of M is contained in arbitrarily small neighborhoods in M with countable boundaries (see K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), p. 277); or, in other words, for each point P of M and each $\epsilon > 0$, P can be ϵ -separated in M by a countable subset of M (see P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, II Verhandelingen der Akademie Amsterdam, vol. 13, No. 4).

is connected and contains $A + B$, this contradicts the assumption that K separates A and B in C); (c) if two points A and B of a continuous curve M do not lie together in a maximal cyclic curve of M , there exists at least *one point* of M which separates A and B in M . Under our hypothesis it follows by (b) and (c) that every two points of M can be separated in M by a countable subset of M ; and hence, by (a), M is a rational curve. Lemma 2 is readily established by constructing, in each complementary domain D of M having the given point P on its boundary, a sequence of arc-crosscuts converging to P such that P is a limit point of no single component of D minus the sum of these crosscuts, and these are added to M to form M^* .

2. *Proof of the Theorem.* Let P be any point of M and ϵ any positive number. By Lemma 1 we may assume M to be cyclicly connected. And by Lemma 2, we may assume, also without loss of generality, that P is on the boundary of no complementary domain of M ; for if this is not the case, we consider the curve M^* and show that M^* is rational at P ; and then since $M^* \supset M$, it follows that M also is rational at P . By a theorem of the author's† there exists a simple closed curve J of diameter $< \epsilon/4$ enclosing P and such that if K denotes the sum of the boundaries of all those complementary domains of M each of which has a boundary point on J , then $J \subset K$ and $\delta(K) < \epsilon/4$. Since K is a subcontinuum of M , it must be a continuous curve. Let C be the maximal cyclic curve of K containing J , and let B be the boundary of the complementary domain of C which contains P . Let $[I_i]$, $i=1, 2, 3, \dots$ be the collection of complementary domains of M such that if, for each i , J_i denotes the boundary of I_i then J_i belongs to K and contains at least one point of B . Then since $B \subset K$ and every point of K is a boundary point of at least one complementary domain of M , therefore $B \subset \sum_1^\infty J_i$. Since C and M are cyclicly connected, therefore‡ each of the curves $B, J_1, J_2, J_3, \dots$ is a simple closed curve. Now since $B \subset M$, it follows that, for each i , the interior I_i of J_i

† Concerning the complementary domains of continua, *Annals of Mathematics*, vol. 29 (1928), pp. 399–411, Theorem 18.

‡ G. T. Whyburn, *Cyclicly connected continuous curves*, *Proceedings of the National Academy of Sciences*, vol. 13 (1927), pp. 31–38, Theorem 10.

lies either wholly within or wholly without B ; and if, for any i , I_i lies within B , then $B \cdot J_i$ consists of exactly one point. Thus if we divide the curves of the collection $[J_i]$ into two classes G_1 and G_2 such that, for each i , J_i belongs to G_1 or to G_2 according as $B \cdot J_i$ consists of a single point or of more than one point, then the interior of every curve of G_2 lies wholly without B .

Now let Q be a point at a distance $\geq \epsilon$ from P . Then Q is without every one of the curves $B, J_1, J_2, \dots$. Let $J_{n_1}, J_{n_2}, \dots$ be those curves of class G_2 each of which contains a point A_{n_i} which is accessible from Q by an arc having only A_{n_i} in common with $\sum J_i$. For each i , let X_i and Y_i be the boundary points of the component of $J_{n_i} - J_{n_i} \cdot B$ which contains A_{n_i} . For each i , $I_{n_i} + X_i + Y_i$ contains an arc T_i from X_i to Y_i . Let H denote the closed point set $\overline{\sum T_i}$. Then clearly $H \subset \sum T_i + B$. Now if E is any point of H not in $\sum T_i$, it is easily seen that E must be accessible from Q by an arc having only E in common with $\sum J_i$. And since E belongs to no arc T_i , clearly it can then belong to no curve of class G_2 . Hence it must belong to a curve of class G_1 ; and thus there exists an integer j such that $E = B \cdot J_j$. Therefore the set of all such points $[E]$ is countable; and since, for each i , $T_i \cdot M = X_i + Y_i$, it follows that $H \cdot M$ is countable.

Now let Z be any point at a distance $\geq \epsilon$ from P . Then Z and Q lie in the same complementary domain of H , since H lies in a circle with center P and radius $\epsilon/2$. However, the set H must separate P and Z . For if not, there exists an arc ZP from Z to P with $H \cdot ZP = Z$. Let V be the first point on ZP , in the order from Z to P , which belongs to B . Since $ZV - V$ contains no point of any of the arcs T_i , it follows that V is accessible from Q and belongs to no curve of class G_2 . But since the set of all such points of B is countable, it is readily seen that every such point is a limit point of $\sum T_i$ and hence belongs to H , contrary to supposition. Thus H ϵ -separates P in the plane, and since $H \cdot M$ is countable it follows that M is rational at P , and thus our theorem is established.

In conclusion I will remark that the above proof depends very sharply upon properties of the plane, and I do not know whether or not the theorem is true in a space of more dimensions, but would be very much interested in a solution for such a space.