

ON CANONICAL FORMS OF DIFFERENTIAL EQUATIONS*

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1. *Introduction.* The projective differential geometry of surfaces has been greatly enriched by the introduction of Fubini's projective normal into the theory. This projective normal however fails to exist for ruled surfaces. In this paper we propose a class of congruences covariantly related to the surface, each congruence being suitable as a substitute for Fubini's projective normal congruence. As an application we give an interesting class of congruences covariantly related to a scroll surface and conjugate to the surface.

If the asymptotic parameters are parametric, the four homogeneous coordinates $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ of a general point on the surface S_v are solutions of differential equations in the Wilczynski semicanonical form:†

$$(1) \quad \begin{cases} y_{uu} + 2ay_u + 2by_v + cy = 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y = 0. \end{cases}$$

One of the integrability conditions of system (1) is

$$(2) \quad a_v = b'_u.$$

If the transformations

$$(3) \quad y = \lambda \bar{y}$$

and

$$(4) \quad \bar{u} = U(u), \quad \bar{v} = V(v)$$

are performed on system (1), the new coefficients are respectively

$$(5) \quad \begin{cases} \bar{a} = a + \frac{\lambda_u}{\lambda}, \quad \bar{b} = b, & \bar{c} = \frac{1}{\lambda}(\lambda_{uu} + 2a\lambda_u + 2b\lambda_v + c\lambda), \\ \bar{a}' = a', \quad \bar{b}' = b' + \frac{\lambda_v}{\lambda}, & \bar{c}' = \frac{1}{\lambda}(\lambda_{vv} + 2a'\lambda_u + 2b'\lambda_v + c'\lambda), \end{cases}$$

* Presented to the Society, April 19, 1930.

† E. J. Wilczynski, *First memoir*, Transactions of this Society, vol. 8 (1907), pp. 233-260.

and

$$(6) \quad \begin{cases} \bar{a} = \frac{1}{U_u} \left(a + \frac{1}{2} \frac{U_{uu}}{U_u} \right), & \bar{b} = \frac{V_v}{U_u^2} b, & \bar{c} = \frac{c}{U_u^2}, \\ a' = \frac{U_u}{V_v^2} a', & \bar{b}' = \frac{1}{V_v} \left(b' + \frac{1}{2} \frac{V_{vv}}{V_v} \right), & \bar{c}' = \frac{c'}{V_v^2}. \end{cases}$$

2. *The Canonical Forms.* Consider an invariant R of the surface with the following properties:

(a) $R \neq 0$, (b) the transform \bar{R} of R by (4) is $\bar{R} = R/(U_u V_v)$.

Condition (2) implies the existence of a function p such that

$$(7) \quad p_u = a + h \frac{R_u}{R}, \quad p_v = b' + h \frac{R_v}{R},$$

where h is a constant. The transforms \bar{p}_u and \bar{p}_v of p_u and p_v by means of (3) are

$$(8) \quad \bar{p}_u = p_u + \frac{\lambda_u}{\lambda}, \quad \bar{p}_v = p_v + \frac{\lambda_v}{\lambda}.$$

If we examine the transforms of p_u and p_v under (4), we observe that these functions will be invariant under that transformation if and only if $h = 1/2$. Hence *the points r and s defined by the formulas*

$$(9) \quad r = y_u + p_u y, \quad s = y_v + p_v y,$$

$$(10) \quad p_u = a + \frac{1}{2} \frac{R_u}{R}, \quad p_v = b' + \frac{1}{2} \frac{R_v}{R},$$

are covariant points. Transformation (3) for $\lambda = e^{-p}$ transforms the expressions r and s into y_u and y_v respectively. With this choice of λ , system (1) assumes the following canonical form:

$$(11) \quad y_{uu} = \theta_u y_u + \beta y_v + P y, \quad y_{vv} = \gamma y_u + \theta_v y_v + Q y,$$

wherein

$$\theta = \log R, \quad \beta = -2b, \quad \gamma = -2a',$$

$$(12) \quad P = -\Phi = -[c - a_u - a^2 - 2bb' - b\theta_v - \frac{1}{2}\theta_{uu} + \frac{1}{4}\theta_u^2],$$

$$Q = -\Psi = -[c' - b_v - b'^2 - 2aa' - a'\theta_u - \frac{1}{2}\theta_{vv} + \frac{1}{4}\theta_v^2].$$

The quadratic form

$$(13) \quad 2Rdu dv$$

is absolutely invariant under (3) and (4). If we compute the second covariant derivatives of y with respect to the form (13), we find that the four functions y satisfy the equations

$$(14) \quad y_{11} = \beta y_2 + P y, \quad y_{22} = \gamma y_1 + Q y,$$

wherein $y_1 = y_u, y_2 = y_v$.

The coordinates y , which satisfy (11) or (14), are similar to Fubini's normal coordinates. In fact if $R = a'b$ the canonical form (14) becomes Fubini's canonical form and the invariants Φ and Ψ become the invariants ϕ and ψ derived by Lane.*

3. *Geometric Considerations.* We shall call the line joining y to the covariant point y_{uv} the R conjugate line, and the line joining the covariant points y_u, y_v the R harmonic line. The congruences generated by these lines are respectively conjugate† and harmonic to the surface.

A geometric characterization of the R conjugate line of course depends upon the particular invariant R . In terms of this characterization, geometric interpretations of the covariants y_u, y_v , and y_{uv} and of the vanishing of the invariants P and Q parallel the work of Lane in the paper just cited.

The developables of the congruence of R conjugate lines intersect S_y in the integral curves of the differential equation

$$(15) \quad (\Phi + 2b_v + 2b\theta_v)du^2 - (\Psi + 2a'_u + 2a'\theta_u)dv^2 = 0.$$

The developables of the congruence of R harmonic lines correspond to the curves defined by the equation

$$(16) \quad \Phi du^2 - \Psi dv^2 = 0.$$

Referred to the tetrahedron y, y_u, y_v, y_{uv} of system (11) the equation of the quadric of Lie is

$$(17) \quad x_1 x_4 - x_2 x_3 + (2a'b + \frac{1}{2}\theta_{uv})x_4^2 = 0.$$

The R conjugate line intersects this quadric in y and in the point defined by

* E. P. Lane, *Wilczynski's and Fubini's canonical systems of differential equations*, this Bulletin, vol. 32 (1926), p. 368.

† G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, Transactions of this Society, vol. 20 (1919), p. 99.

$$(18) \quad y_{uv} - (2a'b + \frac{1}{2}\theta_{uv})y.$$

4. *Applications to Scroll Surfaces.* Suppose now that the surface S_v is a scroll. Let $v = \text{const.}$ be the rulings. It follows therefore that $b = 0$, $a' \neq 0$. From (15) and (16) we observe that, if we take

$$(19) \quad R = \frac{\nu(v)}{a'}$$

where $\nu(v)$ is a function of v alone, the curves corresponding to the developables of the R conjugate and R harmonic congruences coincide. In order to satisfy property (b) of R , the function $\nu(v)$ must be transformed according to the formula

$$(20) \quad \bar{\nu}(v) = \nu(v)/V_v^3.$$

If we let $z = y_u$, we find readily that the four pairs of functions (y, z) are solutions of differential equations of the form

$$\begin{aligned} y_{vv} + p_{11}y_v + p_{12}z_v + q_{11}y + q_{12}z &= 0, \\ z_{vv} + p_{21}y_v + p_{22}z_v + q_{21}y + q_{22}z &= 0, \end{aligned}$$

wherein

$$(21) \quad \begin{aligned} p_{11} &= 2b', & p_{12} &= 0, & q_{11} &= c', & q_{12} &= 2a', \\ p_{21} &= 2b'_u, & p_{22} &= 2b', & q_{21} &= c'_u - 2a'c, & q_{22} &= c' + 2a'_u - 4aa'. \end{aligned}$$

Denote by θ_4 the invariant* whose vanishing implies that the ruled surface has the two branches of its flecnode curve coincident. From (21) we find that θ_4 is expressible in terms of the coefficients of system (1) as follows:

$$(22) \quad \theta_4 = 64[(a'_u - 2aa')^2 - 2a'(b'_{uv} - c'_u + 2a'c + 2b'b'_u)].$$

If we use the integrability conditions† of system (1), it is easily verified that $(\theta_4)_u = 0$. Hence θ_4 is a function of v alone. Moreover from (6) we find that the transform $\bar{\theta}_4$ of θ_4 by (4) is defined by the formula

$$\bar{\theta}_4 = \theta_4/V_v^4.$$

* E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, B. G. Teubner, 1906, p. 104.

† Wilczynski, *First memoir*, p. 245.

We observe that, if the ruled surface has the two branches of its flecnode curve distinct, we may take the function $\nu(v)$ of (19) to be

$$\nu(v) = \theta_4^{3/4}.$$

The R harmonic line, the scroll directrix of the first kind, and the generator intersect in the point y_u for the choice (19) of R . The R conjugate line lies in the plane determined by the scroll directrix of the second kind and the generator.

For the choice (19) of R , and from (18), we see that *the point y_{uv} lies on the osculating quadric for every choice of $\nu(v)$ if and only if*

$$(\log a')_{uv} = 0,$$

that is, if and only if the curved asymptotics on S_y belong to linear complexes. Moreover the point y_u describes a curved asymptotic on S_y as y describes the curve $u = \text{const}$.*

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* C. T. Sullivan, *Properties of ruled surfaces whose asymptotic curves belong to linear complexes*, Transactions of this Society, vol. 15 (1914), p. 190.