AN ELEMENTARY THEOREM ON MATRICES*

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This note applies some elementary algebraic theory to the theory of matrices, securing a generalization of the theorem:

If $y_1, \cdots, y_n$ are elements of a field F corresponding to n distinct elements $x_1, \cdots, x_n$ in F, there exists one and only one polynomial f of degree less than n such that $f(x_i) = y_i (i = 1, \cdots, n)$.

As is well known, every finite square matrix m with elements in a field F satisfies its characteristic equation $|m - \lambda I| = 0$, and hence satisfies a unique equation, $g(\lambda) = 0$, of minimum degree with leading coefficient unity. Moreover, if for two polynomials f and h, $f(m) = h(m)$, then $f(\lambda) \equiv h(\lambda) \bmod g(\lambda)$ and conversely.

We seek an answer to the question "Given finite square matrices $m_1, \cdots, m_n$ and polynomials $h_1, \cdots, h_m$, under what conditions can a polynomial f be found such that $f(m_i) = h_i(m_i)$ $(i = 1, \cdots, n)$?"

If the minimum equations of the m_i are $g_i(\lambda) = 0, (i = 1, \cdots, n)$. the above question is equivalent to asking, under what conditions the congruences

$$f(\lambda) \equiv h_i(\lambda) \qquad \bmod g_i(\lambda), \qquad (i = 1, \cdots, n),$$

have a solution.

Standard works on number theory discuss this question and give its solution in an elementary fashion. In particular, there are always solutions when $g_1, \cdots, g_n$ are relatively prime.

From considerations of this known work on congruences, many theorems, two of which follow, may be translated at once to their matrix form.

THEOREM 1. *If there exists a polynomial f in a field F such that for n finite square matrices $m_1, \cdots, m_n$ with elements in F, and n polynomials $h_1, \cdots, h_n$ in $F_1, f(m_i) = h_i(m_i)$, $(i = 1, \cdots, n)$, then there exists one and only one such f of degree lower than that of k, the least common multiple of the minimum equations of $m_1, \cdots,$*

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m_n, all other solutions being obtained from this solution by the addition of multiples of k.

THEOREM 2. *If the minimum equations of n finite square matrices m_1 to m_n with elements in a field F are relatively prime then for any set of n polynomials $h_1, \cdots, h_m$, in F, a polynomial f may be found such that*

$$f(m_i) = h_i(m_i), \qquad (i = 1, \cdots, n).$$

These theorems specialize to the above mentioned algebraic theorem since each x_i is a one by one matrix with minimum equation $\lambda - x_i = 0$.

It should be noted that in the above discussion no restriction as to the field in which the elements of the matrices may lie is made, nor are the m_i necessarily of the same order.

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PROBLEMS OF THE CALCULUS OF VARIATIONS WITH PRESCRIBED TRANSVERSALITY CONDITIONS*

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1. *Introduction.* Problems of the calculus of variations in the plane for which a prescribed relation exists between the directions of the extremals and the transversals were first studied by Stromquist‡ and Bliss.§ Recently Rawles,|| using a method based on properties of the Hilbert invariant integral, has given an interesting treatment of the analogous problem in $(x, y_1, \cdots, y_n)$-space.

In the present paper the latter problem is attacked from a quite different point of view.¶ The method here used avoids a restrictive hypothesis made by Rawles with regard to the ex-

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‡ Stromquist, Transactions of this Society, vol. 7 (1906), p. 181; Annals of Mathematics, (2), vol. 9 (1907), p. 57.

§ Bliss, Annals of Mathematics, (2), vol. 9 (1907), p. 134.

|| Rawles, Transactions of this Society, vol. 30 (1928), pp. 765–784.

¶ The possibility of approaching the problem from this viewpoint was suggested to the writer by G. A. Bliss.