SINGULAR POINTS OF FUNCTIONS WHICH SATISFY THE PARTIAL DIFFERENTIAL EQUATION OF THE FLOW OF HEAT*

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1. *Introduction.* It has frequently been pointed out that the function

$$U(x, y; a, b) = \frac{1}{(y-b)^{1/2}} e^{\mu}, \qquad (y > b), \tag{1}$$
$$= 0, \qquad (y \leqq b),$$

where $\mu = -(x-a)^2/[4(y-b)]$, plays a rôle in the theory of the solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \tag{2}$$

which is quite analogous to that played by the function

$$\log [(x - a)^2 + (y - b)^2]^{1/2} \tag{3}$$

in the theory of harmonic functions. One would expect, therefore, to find a characterization of the function (1) similar to that given by Bôcher† for (3). It is the purpose of the present note to obtain such a characterization.

For brevity we designate a function as regular in a given region if it is continuous with its first derivatives there. The results to be proved are the following.

THEOREM 1. *If $f(x, y)$ is a single-valued solution of (2) regular in the neighborhood of a point (a, b) except at (a, b), and is bounded, then $f(x, y)$ becomes regular at (a, b) if its definition at this point is properly adjusted.*

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† M. Bôcher, *Singular points of functions which satisfy partial differential equations of the elliptic type*, this Bulletin, vol. 9 (1903), p. 455. See also O. D. Kellogg, *On some theorems of Bôcher concerning isolated singular points of harmonic functions*, this Bulletin, vol. 32 (1926), p. 664. Professor Kellogg has informed the author that Bôcher's principal theorem was also known to Schwarz.

THEOREM 2. *If $f(x, y)$ is a single-valued solution of* (2) *regular in the neighborhood of a point* (a, b) *except at* (a, b), *and if* $f(x, y)$ *is bounded on one side only, then* $f(x, y) - LU(x, y; a, b)$ *is regular at* (a, b) *provided the constant* L *is properly chosen and the function is properly defined at* (a, b).

Our proof, like Bôcher's, employs a Green's function; but the method of proof must necessarily differ considerably from that of Bôcher since the level curves

$$\frac{1}{(y-b)^{1/2}}e^{\mu} = \text{constant}, \qquad \mu = -\frac{(x-a)^2}{4(y-b)},$$

do not in this case consist of closed curves bounding regions with (a, b) in the interior, but all pass through this point.

2. *Green's Formula and Green's Function.* In the present section we shall recall certain results from the classical theory of equation (2). Set

$$F(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}, \qquad G(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\cdot$$

Then $G(u)$ is the adjoint of $F(u)$ and the equation

$$G(u) = 0 \tag{4}$$

is the adjoint of equation (2). Let D be a regular* region with boundary C. If $u(x, y)$ and $v(x, y)$ are two functions continuous with their first and second derivatives in and on the boundary of D, then Green's formula is

$$\int_D\int [vF(u) - uG(v)]dxdy = \int_C uv\,dx + \left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)dy. \tag{5}$$

Here the contour integration is in the positive sense.† We now define the Green's function corresponding to a given rectangle.

DEFINITION. Let the vertices A, B, F, E of a rectangle R have

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* For present purposes the boundary may consist of a finite number of straight-line segments.

† Most of the results of this section will be found in E. Goursat, *Cours d'Analyse*, 1923, vol. 3, Chap. 29. In applying this formula to regular solutions of (2) and (4) we make use of the fact that such solutions are known to have continuous second derivatives.

coördinates (x_0, y_0), (x_0+p, y_0), (x_0+p, y_0+q), (x_0, y_0+q) respectively, p and q being arbitrary positive constants, and let (ξ, η) be an arbitrary interior point. By the Green's function of this rectangle we mean a function $G(x, y; \xi, \eta)$ which, considered as a function of x and y:

(a) satisfies (2) and is regular in the closed region R except at (ξ, η);

(b) has the form

$$G(x, y; \xi, \eta) = U(x, y; \xi, \eta) + u(x, y)$$

at (ξ, η) where $u(x, y)$ satisfies (2) and is regular in the closed region R;

(c) is zero on the line segments EA, AB, BF.

The existence and uniqueness of this function follows from known results concerning a familiar boundary-value problem.* It may also be shown that G satisfies the adjoint equation (4) considered as a function of (ξ, η), and is regular except at the point (x, y), where it has the form

$$U(x, y; \xi, \eta) + v(\xi, \eta),$$

$v(\xi, \eta)$ being a solution of (4) regular throughout R. Moreover, G vanishes on the sides AE, EF, FB when considered as a function of (ξ, η).

3. *Three Lemmas.*

LEMMA 1. *If $\phi(x, t)$ is a solution of* (2) *regular in R except perhaps at an interior point (a, b), then the integral*

$$\int_C \frac{\partial \phi}{\partial x} dy + \phi dx$$

has a constant value K when extended in the clock-wise sense over the contours C of all rectangles lying in R and containing (a, b) as an interior point.

This follows from formula (5) by taking $u=\phi$, $v=1$, and D the region between two of the rectangles considered. In particular, if ϕ is regular at (a, b), K is zero.

LEMMA 2. *If $\phi(x, y)$ satisfies the conditions of Lemma* 1, *is*

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* E. Goursat, loc. cit., p. 316.

bounded at least on one side, and vanishes on the sides EA, AB, BF, then

$$\lim_{h\to 0^+}\int_{a-k}^{a+k}\phi(x, b+h)f(x)dx = Kf(a)$$

for any function $f(x)$ continuous in $(a-k, a+k)$ and for any k such that the points $(a-k, b)$, $(a+k, b)$ lie in R.

PROOF. Since $\phi(x, y)$ is bounded on one side it is possible to find a constant H such that the function $\psi(x, y)=\phi(x, y)+H$ is a function of one sign in R. For definiteness we suppose it positive. As a result of Lemma 1 we have

$$\int_{a-k}^{a+k}\phi(x, b+h)dx + \int_{b}^{b+h}\frac{\partial\phi}{\partial x}(a-k, y)dy + \int_{b+h}^{b}\frac{\partial\phi}{\partial x}(a+k, y)dy = K.$$

We have here chosen the contour C as the boundary of the rectangle whose vertices are $(a-k, b-h), (a+k, b-h), (a+k, b+h)$, $(a-k, b+h)$. It was unnecessary to include the complete contour since ϕ vanishes identically for $y<b$. This follows* since ϕ is zero on the sides EA, AB, BF, and is regular for $y<b$. If h tends to zero we clearly have

$$\lim_{h\to 0^+}\int_{a-k}^{a+k}\phi(x, b+h)dx = K,$$

or

$$\lim_{h\to 0^+}\int_{a-k}^{a+k}\psi(x, b+h)dx = K + 2kH.$$

We shall first show that

$$\lim_{h\to 0^+}\int_{a-k}^{a+k}\psi(x, b+h)f(x)dx = Kf(a) + H\int_{a-k}^{a+k}f(x)dx.$$

The desired result will follow from this equation by noting that

$$\int_{a-k}^{a+k}\phi(x, b+h)f(x)dx = \int_{a-k}^{a+k}\psi(x, b+h)f(x)dx - H\int_{a-k}^{a+k}f(x)dx.$$

* E. Goursat, loc. cit., p. 309.

We employ the following identity:

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\begin{aligned}
I_h &= \int_{a-k}^{a+k} \psi(x, b+h) f(x) dx - K f(a) - H \int_{a-k}^{a+k} f(x) dx \\
&= \int_{a-\alpha}^{a+\alpha} [f(x) - f(a)] \psi(x, b+h) dx + f(a) \int_{a-\alpha}^{a+\alpha} \psi(x, b+h) dx \\
&\quad - K f(a) + \int_{a+\alpha}^{a+k} \phi(x, b+h) f(x) dx \\
&\quad + \int_{a-k}^{a-\alpha} \phi(x, b+h) f(x) dx - H \int_{a-\alpha}^{a+\alpha} f(x) dx.
\end{aligned} \tag{6}
$$

Here α is an arbitrary number for which $0<\alpha<k$. Given an arbitrary positive number ϵ, we determine α so small that the following inequalities hold:*

$$\left| \int_{a-\alpha}^{a+\alpha} f(x) dx \right| < \frac{\epsilon}{5H}, \tag{a}$$

$$| f(x) - f(a) | < \frac{\epsilon}{5(K + 2kH)}, \quad | x - a | \leqq \alpha, \tag{b}$$

$$\alpha < \frac{\epsilon}{10H \, | f(a) |} \cdot \tag{c}$$

With this choice of α equation (6) gives the inequality

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\begin{aligned}
| I_h | \leqq \; & \frac{\epsilon}{5(K + 2kH)} \int_{a-\alpha}^{a+\alpha} \psi(x, b+h) dx \\
& + | f(a) | \left| \int_{a-\alpha}^{a+\alpha} \psi(x, b+h) dx - K \right| \\
& + \left| \int_{a+\alpha}^{a+k} \phi(x, b+h) f(x) dx \right| \\
& + \left| \int_{a+k}^{a-\alpha} \phi(x, b+h) f(x) dx \right| + \frac{\epsilon}{5} \cdot
\end{aligned} \tag{7}
$$

We can now choose h so small that the right-hand side of this

* H may clearly be chosen positive and so great that $K+2kH$ is positive. If $f(a)=0$, condition (c) becomes superfluous.

inequality will be less than ϵ. For we have shown that we can choose h so small that the integral

$$\int_{a-\alpha}^{a+\alpha} \psi(x, b + h)dx$$

differs as little as we like from $K+2\alpha H$, and hence is less than $K+2kH$. For such values of h the first term on the right-hand side of (7) is less than $\epsilon/5$. The second term may be made to differ as little as we like from $2\alpha H|f(a)|$ by choice of h, and hence made less than $\epsilon/5$ by virtue of (c). The two remaining integrals of (7) approach zero as h approaches zero since they are continuous functions of h in the neighborhood of $h=0$ and since $\phi(x, b)=0$ in the intervals $a+\alpha \leqq x \leqq a+k$, $a-k \leqq x \leqq a-\alpha$. We may therefore determine h so that each is in absolute value less than $\epsilon/5$. It follows that $|I_h| < \epsilon$ for h sufficiently small, and the lemma is established.

Lemma 3. *If $\phi(x, y)$ satisfies the conditions of Lemmas 1 and 2, and if $v(x, y)$ is a solution of (4) regular in R, then*

$$(8)\quad \int_{C_0} \phi(x, y)v(x, y)dx + \left[v(x, y)\frac{\partial \phi}{\partial x}(x, y) - \phi(x, y)\frac{\partial v}{\partial x}(x, y)\right]dy = Kv(a, b),$$

where C_0 is the contour of any rectangle in R with its sides parallel to the axes and including (a, b) as an interior point, and where the integration is in the clockwise sense.

Proof. Consider the function $\phi(x, y+h)$ where h is a small positive constant which, in the course of the proof, will be allowed to approach zero as its limit. This function is a solution of (2) regular in the rectangle whose vertices are $(a-k, b)$, $(a+k, b)$, $(a+k, b+r)$, $(a-k, b+r)$ provided r, h and k are chosen positive and sufficiently small; for, the point $(a, b-h)$ at which $\phi(x, y+h)$ may fail to be regular lies outside the rectangle. We may now apply Green's formula to this rectangle taking $u(x, y)=\phi(x, y+h)$ and $v(x, y)=v(x, y)$. Then

$$\int_{a-k}^{a+k} \phi(x, b + h)v(x, b)dx$$

$$= \int_b^{b+r}\left[v(a-k,\ y)\frac{\partial\phi}{\partial x}(a-k,\ y+h)-\phi(a-k,\ y+h)\frac{\partial v}{\partial x}(a-k,\ y)\right]dy$$

$$+ \int_{a-k}^{a+k}\phi(x,\ b+r+h)v(x,\ b+r)dx$$

$$+ \int_{b+r}^{b}\left[v(a+k,\ y)\frac{\partial\phi}{\partial x}(a+k,\ y+h)-\phi(a+k,\ y+h)\frac{\partial v}{\partial x}(a+k,\ y)\right]dy.$$

Now let h approach zero. By Lemma 2 the first integral tends to $Kv(a, b)$. The integrals on the right are continuous functions of h in the neighborhood of $h=0$, so that their limits are obtained by setting $h=0$ in their integrands.

It is now an easy matter to complete the proof. By Green's formula the integral (8) extended over C_0 is equal to the same integral extended over the rectangle whose vertices are $(a-k, b-r)$, $(a+k, b-r)$, $(a+k, b+r)$, $(a-k, b+r)$, where r and k are so chosen that the rectangle lies in R. But we have seen that $\phi(x, y)\equiv 0$ if $y<b$ so that the part of the contour for which $y<b$ may be neglected. The integral extended over the remainder of the contour, as we have just seen, is equal to $Kv(a, b)$. The equation (8) is thus established.

4. *Proof of the Theorems.* Let $f(x,y)$ be a solution of (2) regular in the neighborhood of (a, b) except at (a, b). Choose the rectangle R of Section 2 so that $f(x, y)$ is regular throughout R except at (a, b). Determine a solution $\bar{f}(x, y)$ of (2) regular throughout R and assuming the same values as $f(x, y)$ on the sides EA, AB, BF. If $f(x, y)$ is bounded at least on one side, then the function $\phi(x, y)=f(x, y)-\bar{f}(x, y)$ has the same property, and may be taken as the function ϕ of the lemmas.

Let (x_1, y_1) be an arbitrary interior point of R not the point (a, b). Surround (x_1, y_1) by a small rectangle with boundary C_1, with sides parallel to the axes, lying entirely in R, and such that (a, b) is an exterior point. In a similar way construct a rectangle with boundary C_0 about (a, b) and so small that all points of C_1 are exterior points. Let C be the boundary of R. Form the Green's function $G(x, y; \xi, \eta)$ for the region R. Fix the first pair of arguments at (x_1, y_1), and let the second pair be variable. The functions $\phi(x, y)$ and $G(x_1, y_1; x, y)$ are regular in the region D bounded by the curves C, C_0, C_1, the former a solution of (2), the latter a solution of (4). If we apply Green's formula to these two functions in the region D we obtain

$$\int_{C_0} \phi(x, y)G(x_1, y_1; x, y)dx + \left(G\frac{\partial \phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right)dy$$

$$+ \int_{C_1} \phi(x, y)G(x_1, y_1; x, y)dx + \left(G\frac{\partial \phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right)dy = 0.$$

The integration is in the same sense over C_0 and C_1. The integral extended over C disappears since either ϕ or G vanishes on each side of the rectangle. By Lemma 3, the first of these integrals has the value $\pm KG(x_1, y_1; a, b)$. The second may easily be evaluated by known theory. For, we recall that, in the neighborhood of (x_1, y_1), $G(x_1, y_1; x, y) = U(x_1, y_1; x, y) + v(x, y)$, where $v(x, y)$ is a solution of (4) regular in the region bounded by C_1. Moreover.

$$\int_{C_1} \phi v dx + \left(v\frac{\partial \phi}{\partial x} - \phi\frac{\partial v}{\partial x}\right)dy = 0,$$

and

$$\int_{C_1} \phi(x, y)U(x_1, y_1; x, y)dx + \left(-\phi\frac{\partial U}{\partial x} + U\frac{\partial \phi}{\partial x}\right)dy$$
$$= \pm 2\pi^{1/2}\phi(x_1, y_1).*$$

Hence $2\pi^{1/2}\phi(x_1, y_1) = \pm KG(x_1, y_1; a, b)$. As a function of (x_1, y_1), G is a solution of (2), and we have

$$\phi(x, y) = \pm \frac{K}{2\pi^{1/2}}U(x, y; a, b) + u(x, y),$$

$$f(x, y) = \pm \frac{K}{2\pi^{1/2}}U(x, y; a, b) + u(x, y) + j(x, y). \tag{9}$$

We see that if $f(x, y)$ is bounded on both sides K must be zero, for, otherwise the right-hand side of (9) would be unbounded, and we should have a contradiction. On the other hand, if $f(x, y)$ is unbounded on one side, K can not be zero. If we choose the constant L of Theorem 2 as $\pm K/(2\pi^{1/2})$, equation (9) serves to prove that theorem. Both theorems are thus completely established.

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*E. Goursat, loc. cit., p. 311.