CONCERNING A SET OF AXIOMS FOR THE SEMI-QUADRATIC GEOMETRY OF A THREE-SPACE*

BY J. L. DORROH

In his paper *Sets of metrical hypotheses for geometry*,† R. L. Moore raises the question whether the set O of order axioms and the set C of congruence axioms employed therein, together with M, the proposition that every segment has a mid-point, and P_2, a form of the parallel postulate, are sufficient to give the semi-quadratic geometry of a three-space. At the same time, he states that this question may be answered in the affirmative if it can be proved on the basis of O, C, and M that all right angles in space are congruent to each other. In the present paper it will be shown that O and C are sufficient to require that all right angles in space be congruent to each other.

It is a result of a recent paper‡ of the present author that the theorems of sections 1, 2, 3, and 4 of M.H. are consequences of O and C. Theorems from these sections of M.H. will be quoted without further mention of this justification of their use.

THEOREM 1. *If A, B, C, D are four non-coplanar points such that $\measuredangle ABD$ is a right angle§ and $\measuredangle CBD$ is a right angle, and E is any point distinct from B and in the plane ABC, then $\measuredangle EBD$ is a right angle.*

PROOF. If E is a point of the line AB, or of the line CB, then, by hypothesis, $\measuredangle EBD$ is a right angle.

Suppose, then, that E belongs to the plane ABC, is distinct from B, and belongs neither to the line AB nor to the line BC. Let C' denote a point such that CBC'. It follows by a corollary

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† Transactions of this Society, vol. 9 (1908), pp. 487–512. The notation M. H. will be used to designate this paper. Similarly, S. A. will be used to denote O. Veblen's paper, *A system of axioms for geometry*, ibid., vol. 5 (1904), pp. 343–384.

‡ *Concerning a set of metrical hypotheses for geometry*, Annals of Mathematics, (2), vol. 29 (1928), pp. 229–231.

§ See Definition 7 of M. H., §3.

of Theorem 16 of S.A. that the line BE contains a point H such that AHC or AHC'. Let G denote one of the points C or C' so that AHG. Let F denote a point such that DBF and $DB \equiv BF$. Since by hypothesis the line BD is perpendicular to the line AB and to the line BG, it follows that $DG \equiv FG$ and $AD \equiv AF$. Since $AG \equiv AG$ and $AH \equiv AH$, it follows* that $DH \equiv FH$. Hence, by definition, $\sphericalangle DBH$ is a right angle.

THEOREM 2. *If L, M, N, O are four non-coplanar points such that $\sphericalangle LON$ is a right angle and $\sphericalangle MON$ is a right angle, then $\sphericalangle LON \equiv \sphericalangle MON$.*

PROOF. Since L, M, N, O are non-coplanar, L, O, M are non-collinear. Let E denote a point such that the ray OE bisects $\sphericalangle LOM$.† Let M' denote a point in the order MOM', and let Q denote a point such that the ray OQ bisects $\sphericalangle M'OL$. Then $\sphericalangle EOQ$ is a right angle.‡ Let P denote a point such that QOP and $OP \equiv OQ$; then $QE \equiv PE$. Also, since by Theorem 1 $\sphericalangle NOP \equiv \sphericalangle NOQ$, $QN \equiv PN$. The ray OM contains a point K such that PKE, and the ray OL contains a point R such that QRE. By Theorem 1 of M.H. §3, $OK \equiv OR$ and $EK \equiv ER$. Since PKE, QRE, $EP \equiv EQ$, $NE \equiv NE$, $NP \equiv NQ$, and $EK \equiv ER$, then $NK \equiv NR$,§ and, by definition, $\sphericalangle NOR \equiv \sphericalangle NOK$.

THEOREM 3. *If α_1 and α_2 are two intersecting planes and ϕ_1 is a right angle in α_1 and ϕ_2 is a right angle in α_2, then $\phi_1 \equiv \phi_2$.*

PROOF. Let k denote the line of intersection‖ of α_1 and α_2. Let k_1 denote a line in α_1 perpendicular to k at a point O of k, and let k_2 denote a line in α_2 perpendicular to k at O. Let ψ_1 be a right angle formed by k_1 and k, and let ψ_2 be a right angle formed by k_2 and k. It follows from Theorem 2 that $\psi_1 \equiv \psi_2$.

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* A special case of Theorem 11 of M. H. §1 may be stated as follows: *If A, B, C are three non-collinear points and A', B', C' are three non-collinear points, and ADC, $A'D'C'$, $AB \equiv A'B'$, $AC \equiv A'C'$, $AD \equiv A'D'$, $BC \equiv B'C'$, then $BD \equiv B'D'$.* For the suggestion that the figure used in the proof of Theorem 1 and the use of the particular theorem just stated would shorten the arguments I had previously given for Theorems 1 and 2, I am indebted to H. G. Forder.

† See a corollary of Theorem 6 of M. H., §3.

‡ See proof of Theorem 7 of M. H., §3.

§ See the theorem stated in a footnote on Theorem 1.

‖ See Theorem 25 of S. A., p. 363.

By Theorem 1 of M.H. §4, $\phi_1 \equiv \psi_1$, and $\phi_2 \equiv \psi_2$. It follows, then, from Theorem 14 of M.H. §1, that $\phi_1 \equiv \phi_2$.

THEOREM 4. *If ϕ_1 and ϕ_2 are two right angles in space, then $\phi_1 \equiv \phi_2$.*

PROOF. If ϕ_1 and ϕ_2 are in the same plane, $\phi_1 \equiv \phi_2$ by Theorem 1 of M.H. §4. If ϕ_1 and ϕ_2 are not in the same plane, they lie in intersecting planes or in non-intersecting planes. If they lie in intersecting planes, they are congruent to each other by Theorem 3. If ϕ_1 and ϕ_2 lie in the planes α_1 and α_2, respectively, and α_1 does not intersect α_2, there exists a plane α_3 which intersects both α_1 and α_2. There exists in α_3 a right angle ϕ_3. By Theorem 3, $\phi_1 \equiv \phi_3$ and $\phi_2 \equiv \phi_3$; hence, by Theorem 14 of M.H §1, we have $\phi_1 \equiv \phi_2$.

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CERTAIN QUINARY FORMS RELATED TO THE SUM OF FIVE SQUARES*

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1. *Introduction.* The number of solutions in integers x, y, z of the equation $n = x^2 + y^2 + z^2$ is a function of the binary class number of n. For numerous forms $f = ax^2 + by^2 + cz^2$, the expression of the number of solutions of $f = n$ in terms of the class number is another way of showing that the number of representations of n by f is a function of the number of representations of various multiples of n as the sum of three squares.‡

Similarly, the number of solutions of the equation $n = x^2 + y^2 + z^2 + t^2$ in integers is the sum of the positive odd divisors of n, multiplied by 8 or 24, according as n is odd or even. There are various forms $f = ax^2 + by^2 + cz^2 + dt^2$ for which the number of representations of n by f is a multiple of the sum of the odd divisors of n. The number of representations of n by one of

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‡ See, for example, Kronecker, Journal für Mathematik, vol. 57 (1860), p. 253; J. V. Uspensky, American Journal of Mathematics, vol. 51 (1929), p. 51; B. W. Jones, American Mathematical Monthly, vol. 36 (1929), p. 73.