By Theorem 1 of M.H. §4, $\phi_1 \equiv \psi_1$, and $\phi_2 \equiv \psi_2$. It follows, then, from Theorem 14 of M.H. §1, that $\phi_1 \equiv \phi_2$.

THEOREM 4. *If ϕ_1 and ϕ_2 are two right angles in space, then* $\phi_1 \equiv \phi_2$.

PROOF. If ϕ_1 and ϕ_2 are in the same plane, $\phi_1 \equiv \phi_2$ by Theorem 1 of M.H. §4. If ϕ_1 and ϕ_2 are not in the same plane, they lie in intersecting planes or in non-intersecting planes. If they lie in intersecting planes, they are congruent to each other by Theorem 3. If ϕ_1 and ϕ_2 lie in the planes α_1 and α_2, respectively, and α_1 does not intersect α_2, there exists a plane α_3 which intersects both α_1 and α_2. There exists in α_3 a right angle ϕ_3. By Theorem 3, $\phi_1 \equiv \phi_3$ and $\phi_2 \equiv \phi_3$; hence, by Theorem 14 of M.H §1, we have $\phi_1 \equiv \phi_2$.

THE UNIVERSITY OF TEXAS

CERTAIN QUINARY FORMS RELATED TO THE SUM OF FIVE SQUARES*

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1. *Introduction.* The number of solutions in integers x, y, z of the equation $n = x^2 + y^2 + z^2$ is a function of the binary class number of n. For numerous forms $f = ax^2 + by^2 + cz^2$, the expression of the number of solutions of $f = n$ in terms of the class number is another way of showing that the number of representations of n by f is a function of the number of representations of various multiples of n as the sum of three squares.‡

Similarly, the number of solutions of the equation $n = x^2 + y^2 + z^2 + t^2$ in integers is the sum of the positive odd divisors of n, multiplied by 8 or 24, according as n is odd or even. There are various forms $f = ax^2 + by^2 + cz^2 + dt^2$ for which the number of representations of n by f is a multiple of the sum of the odd divisors of n. The number of representations of n by one of

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‡ See, for example, Kronecker, Journal für Mathematik, vol. 57 (1860), p. 253; J. V. Uspensky, American Journal of Mathematics, vol. 51 (1929), p. 51; B. W. Jones, American Mathematical Monthly, vol. 36 (1929), p. 73.

these forms is thus a simple function of the number of representations of n as the sum of four squares.*

Following a suggestion of E. T. Bell, I have here considered as the fundamental function, $\phi(n)$, the number of representations of n as the sum of five squares. With the exception of three forms (f_{11}, f_{12}, f_{13}), the number of solutions of $n=x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2$, where $a_i=1$, 2 or 4, is shown to be expressible in terms of ϕ and for $a_i=1, 2, 4$, or 8 the number of solutions of $f=n$ is expressed in terms of ϕ and two other functions (α and β). It should be noted that for certain values of n, $M_{abc}(n)$ is expressible *totally* in terms of ϕ. It is true in many cases when n is even and in the following formulas when n is odd: (37.2), (37.3), (38.1), (40.1), (40.2), (41.1), (44.1), (46.1), (48.1), (48.2), (50.1), (52.1). See also the last section of the paper giving a few miscellaneous results.

2. *Notations.* The letters n, m, x, y, μ are used to denote integers; m and μ are odd and n and m are positive.

$N[n=f]$ denotes the number of representations of n by the form $f=x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2$, the coefficients to be arranged in *increasing order of magnitude.*

f_j or f_j' is the form f when j of the coefficients are 2 or 4 respectively and the rest of the coefficients are 1.

f_{ij} is the form f when i of the coefficients are 2, j of them 4 and the remainder are 1.

f_{abc} is the form $x_1^2+ax_2^2+bx_3^2+cx_4^2+8x_5^2$ where a, b and c are powers of 2.

$M_j(n)=N[n=f_j]$; $M_j'(n)=N[n=f_j']$; $M_{ij}(n)=N[n=f_{ij}]$; $M_{abc}(n)=N[n=f_{abc}]$.

We regard the following as fundamental functions:
$M_0(n)=\phi(n)$; $\alpha(m)=M_{11}(m)$; $\beta(m)=M_{244}(m)$, if $m\equiv 1 \pmod 8$.

We also use the following for brevity's sake: $\lambda(n)=M_1'(n)$; $\lambda'(4n)=N[4n=f_1'; x_1x_2x_3x_4 \text{ odd}]$; $\alpha'(n)=N[n=f_{11} \text{ with } x_1 \text{ odd}]$ and $\phi'(m)=N[m=f_0; x_1x_2x_3x_4x_5 \text{ odd}]$, which has a value different from 0 only when $m\equiv 5 \pmod 8$.

3. *A Fundamental Lemma.*† $N[2n=x^2+y^2]=N[n=x^2+y^2]$.

* See, for example, J. Liouville, Journal de Mathématiques, (2), vol. 7 (1862); P. Pepin, Journal de Mathématiques, (4), vol. 6 (1890), p. 5.

† Since this lemma is very elementary, we shall use it freely without comment.

This follows from the fact that $2n=x^2+y^2$ implies that the pair of equations $x+y=2X$ and $x-y=2Y$ is solvable for X and Y and there is a one to one correspondence between the solutions of $2n=x^2+y^2$ and $n=X^2+Y^2$.

COROLLARY 1.

$$N[2m = x^2 + y^2] = 2N[m = \mu^2 + 4y^2].$$

COROLLARY 2.

$$N[2n' = x^2 + y^2] = N[n' = x^2 + y^2] = N[2n' = 2x^2 + 2y^2].$$

4. *Reduction Formulas for* $\phi(n)$. Since $f_0=n\equiv 0 \pmod 4$ implies that just one or all of the x's are even, we have

$$(1) \qquad \phi(4n) - \phi(n) = 5\lambda'(4n).$$

Applying Corollary 1, we have

$$\begin{aligned}\lambda'(4n) &= 4N[2n = \mu_1^2 + \mu_2^2 + 4x_3^2 + 4x_4^2 + 2x_5^2] \\ &= 8N[n = \mu_1^2 + 4x_2^2 + 2x_3^2 + 2x_4^2 + x_5^2].\end{aligned}$$

Applying Corollary 2, we have

$$(1')\ \lambda'(4n) = 8N[n = \mu_1^2 + 4x_2^2 + x^2 + y^2 + x_5^2;\ x \equiv y \pmod 2],$$

$$(2)\ \lambda'(4n) = 8\lambda'(n) \text{ if } n \equiv 0 \pmod 4.$$

If $n=2m$ note that $f_0=2m$ implies that exactly two x's are odd and we have from (1′)

$$(3) \qquad \lambda'(8m) = 4\phi(2m)/5.$$

If $n=m$ consider first the case $m\equiv 5 \pmod 8$. Then

$$\begin{aligned}\phi'(m) &= 4N[m = f_{228} \text{ with } x_1x_2x_3 \text{ odd}] \\ &= 8N[m = \mu_1^2 + 4\mu_2^2 + 8x_3^2 + 16x_4^2 + 8x_5^2].\end{aligned}$$

Now $f_4' = m$ implies that one of x_2, x_3, x_4, x_5 is incongruent mod 2 to the other three, and thus

$$\begin{aligned}M_4'(m) &= 4N[m = f_4';\ x_2 \not\equiv x_3 \equiv x_4 \equiv x_5 \pmod 2] \\ &= 4N[m = \mu_1^2 + 4\mu_2^2 + 16x_3^2 + 8x_4^2 + 8x_5^2;\ x_4 \equiv x_5 \pmod 2] \\ &\quad + 4N[m = \mu_1^2 + 16x_2^2 + 4\mu_3^2 + 8x_4^2 + 8x_5^2;\ x_4 \not\equiv x_5 \pmod 2] \\ &= 4N[m = \mu_1^2 + 4\mu_2^2 + 8x_3^2 + 16x_4^2 + 8x_5^2].\end{aligned}$$

Thus $M_4'(m)=\phi'(m)/2$. This taken with the known equation

$5M_4'(m)+\phi'(m)=\phi(m)$, found by noting that $f_0=m$ implies that just one or all the x's are odd, gives

$$(4) \qquad \phi'(m)2\phi(m)/7 \quad \text{if} \quad m \equiv 5 \pmod 8,$$

and

$$(5) \qquad M_4'(m) = \phi(m)/7 \quad \text{if} \quad m \equiv 5 \pmod 8.$$

Now from (1′), we have $\lambda'(4m)=8M_4'(m)$ if $m\equiv 5 \pmod 8$. Since $f_0=m$ implies that just three of the x's are odd, or just one is odd according as $m\equiv 3 \pmod 4$ or $\equiv 1 \pmod 8$, we have, from (1′) and (5),

$$(6) \qquad \lambda'(4m) = 8a\phi(m),$$

where $a=1/10$, $1/5$, or $1/7$, according as

$$m \equiv 3 \pmod 4, \equiv 1 \pmod 8, \quad \text{or} \quad \equiv 5 \pmod 8.$$

If $n'\not\equiv 0 \pmod 4$, we have, using (1) and (2),

$$\phi(4^{\alpha}n') - \phi(4^{\alpha-1}n') = 5\cdot 8^{\alpha-1}\lambda'(4n'), \ (\alpha \geqq 1),$$
$$\phi(4^{\alpha-1}n') - \phi(4^{\alpha-2}n') = 5\cdot 8^{\alpha-2}\lambda'(4n'), \ (\alpha \geqq 2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\phi(4n') - \phi(n') = 5\lambda'(4n').$$

Adding, we get

$$(7) \qquad \phi(4^{\alpha}n') - \phi(n') = 5\frac{8^{\alpha}-1}{7}\lambda'(4n'),$$

where $\alpha\geqq 1$ and $n_3\not\equiv 0 \pmod 4$. Then, using (3) and (6), we have the reduction formulas

$$(8.1) \qquad \phi(2^{2\alpha+1}m) = (2^{3\alpha+2}+3)\phi(2m)/7,$$

for $\alpha\geqq 1$,

$$(8.2) \qquad \phi(4^{\alpha}m) = A\phi(m)/7,$$

where $\alpha\geqq 1$ and $A=2^{3\alpha+2}+3$, $8^{\alpha+1}-1$, $(5\cdot 8^{\alpha+2}+9)/7$ according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$.

5. *Relationship between* $\lambda(n)$ *and* $\phi(n)$. It is obvious that $\lambda'(4n)=\lambda(4n)-\phi(n)$. Thus, from (1), we have $\phi(4n)+4\phi(n)=5\lambda(4n)$. This from (8) gives the formulas below for $\lambda(4n)$. We

find $\lambda(m)$ by noting that $f_0=m$ implies that just three of the x's are odd or just one is odd according as $m\equiv 3 \pmod 4$ or $m\equiv 1 \pmod 8$ and that $\lambda(m)=4M_4'(m)$ if $m\equiv 5 \pmod 8$. To obtain $\lambda(2m)$ note that $f_0=2m$ implies just two x's are odd.

$$(9.1)\qquad \lambda(4^{\alpha+1}m) = B\phi(m),$$

where $\alpha\geqq 0$ and $B=3(2^{3\alpha+4}+5)/35$, $(3\cdot 2^{3\alpha+5}-5)/35$, or $3(2^{3\alpha+5}+3)/49$, according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$;

$$(9.2)\qquad \lambda(4^{\alpha}\cdot 2m) = 3(2^{3\alpha+1}+5)\phi(2m)/35,$$

where $\alpha\geqq 0$;

$$(9.3)\qquad \lambda(m) = 4a\phi(m),$$

where a is defined in (6).

6. *Forms f, where $M(n)$ is Expressible Totally in Terms of ϕ.**

CASE I: $n=m$. Note that $\lambda(2m)=6N[f_1'=2m;\ x_1x_2$ odd and both x_3 and x_4 even$]=12\alpha'(m)$. Thus

$$(10)\qquad \alpha'(m) = \phi(2m)/20.$$

Now $f_1=m$ implies that one of x_1, x_2, x_3, x_4 is incongruent to the other three modulo 2, that is,

$$M_1(m) = 4N[m = f_1;\ x_1 \not\equiv x_2 \equiv x_3 \equiv x_4 \pmod 2] = 4\alpha'(m)$$

and

$$(11)\qquad M_1(m) = \phi(2m)/5.$$

The equation $f_1'=m$ implies that just three of x_1, x_2, x_3, x_4 are odd, or just one is odd, according as $m\equiv 3$ or $1 \pmod 4$. Thus, using (9.3),

$$(12)\qquad M_2'(m) = b\phi(m),\quad \text{where } b = 1/10,\ 3/5,\ 3/7,$$

according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$.

We note that

$$M_4(m) = N[m = f_0;\ x_1 \text{ odd},\ x_2\equiv x_3,\ x_4\equiv x_5 \pmod 2)].$$

It is therefore true that $M_4(m)=2M_2'(m)$, $M_2'(m)/3$ or

* For complete results see case II below.

$M_2'(m)/3+\phi'(m)$, according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$. Thus, from (12) and (4),

$$(13) \qquad M_4(m) = c\phi(m),$$

where $c=3/7$ or $1/5$ according as $m\equiv 5 \pmod 8$ or $\not\equiv 5 \pmod 8$.

Also $m=f_2'$ implies that all of x_1, x_2, x_3 are odd, or just one is odd according as $m\equiv 3$ or $1 \pmod 4$ and thus $M_{22}(m)=M_2'(m)$ or $M_2'(m)/3$ respectively. We have

$$(14) \qquad M_{22}(m) = a\phi(m),$$

where a is defined in (6). Since $f_2=m$ implies that just one of x_1, x_2, x_3 is odd, or all are odd, we see that

$$M_2(m) = N'(m) + 3M_{22}(m),$$

where $N'(m)=N[m=f_2;\ x_1x_2x_3 \text{ odd}]$. Now

$$N'(m) = M_2'(m),\ 0,\ \text{or } \phi'(m),$$

according as $m\equiv 3 \pmod 4$, $\equiv 1$, or $5 \pmod 8$. Using (12), (14) and (4), we have

$$(15) \qquad M_2(m) = d\phi(m),$$

where $d=2/5$, $3/5$, or $5/7$, according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$.

The following results are obvious:

$$(16) \qquad M_{31}(m) = \alpha'(m) = \phi(2m)/20;$$

$$(17) \qquad M_3(m) = 2M_{31}(m) = \phi(2m)/10;$$

$$(18.1) \qquad M_3'(m) = 2M_2'(m)/3 = 2\phi(m)/5 \text{ or } 2\phi(m)/7,$$

according as $m\equiv 1$ or $5 \pmod 8$;

$$(18.2) \qquad M_3'(4n+3) = 0;$$

$$(19) \qquad M_4'(m) = \tfrac{1}{2}M_3'(m) = e\phi(m),$$

where $e=0$, $1/5$, or $1/7$ according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$;

$$(20) \qquad M_{21}(m) = 2M_{22}(m) = 2a\phi(m),$$

where a is defined in (6).

Case II: n even. We express $N[n=f]$ in terms of λ and ϕ from which, by reference to formulas (9) and (8), $N[n=f]$ may be expressed in terms of ϕ alone. Since $f_1' = 4n$ implies $x_1 \equiv x_2 \equiv x_3 \equiv x_4 \pmod 2$,

$$(21) \qquad M_1(2n) = \lambda(4n), \qquad (\text{for } M_1(m) \text{ see (11)}),$$

$$(22) \qquad M_3(2n) = \phi(n), \qquad (\text{for } M_3(m) \text{ see (17)}).$$

Since

$$\phi(2n) = N[4n = f_3;\ x_1 \equiv x_2 \pmod 2)] = M_2(2n) + 2N'(2n),$$

where $N'(2n) = N[2n = f_1'\,;\ x_1 \text{ odd}]$, we see that

$$N'(2m) = 3N[2m = f_3'\,] = 6M_{31}(m),$$

and that

$$N'(4n) = \lambda'(4n).$$

Using (16) and (1), we then have

$$(23.1) \quad M_2(2m) = 2\phi(2m)/5, \ (\text{for } M_2(m) \text{ see (15)}),$$

$$(23.2) \quad M_2(4n) = \phi(4n) - 2\lambda'(4n) = \{3\phi(4n) + 2\phi(n)\}/5.$$

Obviously,

$$(24) \qquad M_4(2n) = M_1(n) = \phi(2n)/5 \text{ or } \lambda(2n),$$

according as n is odd or even. (For $M_4\,(m)$ see (13).)

Now $M_2'\,(2m) = 3N[2m = f_3'\,] = 6M_{31}(m)$. Thus, using (16), we get

$$(25.1) \qquad M_2'(2m) = 3\phi(2m)/10.$$

For $M_2'\,(m)$, see (12).

$$(25.2) \qquad M_2'(4n) = \phi(n).$$

Also $M_1'\,(2n) = M_3(n)$, using (17),

$$(26) \qquad M_3'(2n) = \phi(2n)/10 \text{ or } \phi(n/2),$$

according as n is odd or even. (For $M_3'\,(m)$, see (18).)

The following results are obvious:

$$(27) \qquad M_4'(2n) = 0 \text{ or } \phi(n/2),$$

according as n is odd or even. (For $M_4'\,(m)$ see (19).)

$$(28) \qquad M_{21}(2n) = M_1(n) = \phi(2n)/5 \text{ or } \lambda(2n),$$

according as n is odd or even. (For $M_{21}(m)$, see (20).)

(29) $$M_{22}(2n) = M_3(n) = \phi(2n)/10 \text{ or } \phi(n/2),$$

according as n is odd or even. (For $M_{22}(m)$, see (14).)

(30.1) $$M_{31}(2m) = M_2(m) = d\phi(m),$$

where d is defined in (15).

(30.2) $$M_{31}(4n) = 2\phi(2n)/5 \text{ or } \{3\phi(2n) + 2\phi(n/2)\}/5,$$

according as n is odd or even. (For $M_{31}(m)$, see (16).)

7. *A Reduction Formula for* $\alpha'(n)$. A reduction formula for $\alpha'(n)$ will later be found nceessary. We see that

$$\alpha'(2m) = 2N[2m = \mu_1^2 + \mu_2^2 + 8x_3^2 + 4x_4^2 + 4x_5^2] = 4M_{22}(m)$$

and

$$\alpha'(4n) = 2N[2n = \mu_1^2 + \mu_2^2 + 2\mu_3^2 + 4x_4^2 + 4x_5^2]$$
$$= 4N[2n = f_{21};\ x_1x_2 \text{ odd}] = 8\alpha'(n).$$

Thus, using (10), we have

(31.1) $$\alpha'(4^\alpha m) = 8^\alpha\phi(2m)/20,\ \alpha \geqq 0,$$

(31.2) $$\alpha'(4^\alpha\cdot 2m) = 4\cdot 8^\alpha a\phi(m),$$

where a is defined in (6) and $\alpha \geqq 0$.

8. M_{11}, M_{13}, M_{12} *Expressed in Terms of* α *and* ϕ.* It is clear that $M_{11}(2n) = 6N'(n) + M_4(n)$ where $N'(n) = N[n = f_{21};\ x_1 \text{ odd}] = M_{22}(n)$ or $2\alpha'(n/2)$ according as n is odd or even, and

(32.1) $$M_{11}(m) = \alpha(m), \quad \text{by definition,}$$

(32.2) $$M_{11}(2m) = g\phi(m), \quad \text{where } g = 4/5,\ 7/5, \text{ or } 9/7,$$

according as $m \equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$;

(32.3) $$M_{11}(4n) = 12\alpha'(n) + M_4(2n).$$

Now

$$\alpha(m) = N[m = f_{11};\ x_2 \equiv x_3 \pmod 2]$$
$$+ 2N[m = f_{12};\ x_2 \equiv 1 \pmod 2] = \alpha'(m) + 2M_{13}(m),$$

* In many cases, to save space, results are expressed in terms of M_i, M_i', M_{21}, M_{22}, M_{31}, α' which have been previously expressed in terms of ϕ.

(33.1) $M_{13}(m) = \frac{1}{2}\{\alpha(m) - \alpha'(m)\} = \alpha(m)/2 - \phi(2m)/40,$

(33.2) $M_{13}(2n) = M_4(n),$

(34.1) $M_{12}(m) = 2M_{13}(m) = \alpha(m) - \phi(2m)/20,$

(34.2) $M_{12}(2n) = M_2(n).$

9. *M_{abc} Expressed in Terms of α, β' and ϕ.**

If $n = m \equiv 1 \pmod 4$,

$$M_{13}(m) = M_{444}(m) = M_{122}(m)/2.$$

If $n = m \equiv 3 \pmod 4$,

$$\begin{aligned} M_1(m) &= 4N[m = f_{11};\ x_1x_2x_3 \text{ odd}] + 4M_{13}(m) \\ &= 8N[m = f_{224};\ x_1x_2 \text{ odd}] + 4M_{13}(m) \\ &= 8M_{248}(m) + 4M_{13}(m) = 2M_{122}(m) + 4M_{13}(m). \end{aligned}$$

(35.1) $M_{122}(m) = \alpha(m) - \phi(2m)/20$ if $m \equiv 1 \pmod 4$,

(35.2) $M_{122}(m) = 3\phi(2m)/20 - \alpha(m)$ if $m \equiv 3 \pmod 4$,

(35.3) $M_{122}(2n) = \lambda(n).$

Noting that $M_{111}(m) = 4M_{224}(m) = 2M_{122}(m)$, we find

(36.1) $$M_{111}(m) = 2\alpha(m) - \phi(2m)/10,$$

if $m \equiv 1 \pmod 4$;

(36.2) $$M_{111}(m) = 3\phi(2m)/10 - 2\alpha(m),$$

if $m \equiv 3 \pmod 4$;

(36.3) $$M_{111}(2m) = 6M_{21}(m) = 12a\phi(m),$$

where a is defined in (6);

(36.4) $$M_{111}(4n) = \lambda(2n).$$

If $m \equiv 5 \pmod 8$,

$$M_{448}(m) = N[m = f_4';\ x_4 \equiv x_5 \pmod 2]$$

and

$$\begin{aligned} &4N[m = f_4';\ x_2 \not\equiv x_3 \equiv x_4 \equiv x_5 \pmod 2] \\ &\qquad = 4N[m = x_1^2 + 4\mu_2^2 + 16x_3^2 + 4x_4^2 + 4x_5^2] = 2M_{244}(m). \end{aligned}$$

* See note on §8.

Then

$$(37.1) \qquad M_{244}(m) = \beta(m),$$

if $m \equiv 1 \pmod 8$ by definition;

$$(37.2) \qquad M_{244}(m) = M_4'(m)/2 = \phi(m)/14.$$

if $m \equiv 5 \pmod 8$;

$$(37.3) \qquad M_{244}(m) = M_{22}(m)/2 = \phi(m)/20$$

if $m \equiv 3 \pmod 4$;

$$(37.4) \qquad M_{244}(2n) = M_{31}(n).$$

It is clear that

$$M_{112}(m) = 3M_{244}(m) + N'(m),$$

where $N'(m) = N[m = f_{112};\ x_1x_2x_3 \text{ odd}]$.

Now

$$N'(m) = 2N[m = \mu_1^2 + 2\mu_2^2 + 8x_3^2 + 2x_4^2 + 8x_5^2] = M_{22}(m),$$

$\phi'(m)/2$, or 0, according as $m \equiv 3$, 5, or $\pm 1 \pmod 8$. Moreover,

$$M_{112}(2n) = M_{31}(n) + 3N[2n = f_{124};\ x_1x_2 \text{ odd}] = M_{31}(n) + 6N''(n)$$

where $N''(n) = N[n = f_{12};\ x_1 \text{ odd}]$. Now $N''(n) = M_{13}(n)$ if n is odd, $2M_{22}(n/2)$ if $n \equiv 2 \pmod 4$, or $4\alpha'(n/4)$ if $n \equiv 0 \pmod 4$.

$$(38.1) \quad M_{112}(m) = 3\beta(m), \quad 5\phi(m)/14, \quad \phi(m)/4, \quad \text{or} \quad 3\phi(m)/20,$$

according as $m \equiv 1$, 5, 3, or 7 (mod 8);

$$(38.2) \qquad M_{112}(2m) = 3\alpha(m) - \phi(2m)/10;$$

$$(38.3) \qquad M_{112}(4m) = (d + 12a)\phi(m),$$

where d and a are defined in (15) and (6), respectively;

$$(38.4) \qquad M_{112}(8n) = M_{31}(4n) + 24\alpha'(n);$$

$$(39.1) \quad M_{114}(m) = 3M_{111}(m)/4 = 3\alpha(m)/2 - 3\phi(2m)/40 \text{ if } m \equiv 1 \pmod 4;$$

$$(39.2) \quad M_{114}(m) = M_{111}(m)/4 = 3\phi(2m)/40 - \alpha(m)/2 \text{ if } m \equiv 3 \pmod 4;$$

$$(39.3) \quad M_{114}(2m) = 3M_{21}(m) = 6a\phi(m),$$

where a is defined in (6);

(39.4) $$M_{114}(4n) = M_1(n).$$

Noting that $M_{118}(m)=2N[m=f_{288};\ x_2 \text{ odd}]+3M_{448}(m)$, we have

(40.1) $$M_{118}(m) = 2M_{244}(m) = \phi(m)/10,$$

if $m\equiv 3 \pmod 8$;

(40.2) $$M_{118}(m) = 3M_{244}(m) = 3\beta(m) \text{ or } 3\phi(m)/14,$$

according as $m\equiv 1$ or $5 \pmod 8$;

(40.3) $$M_{118}(m) = 0 \text{ if } m \equiv 7 \pmod 8;$$

(40.4) $$M_{118}(2m) = 3M_{12}(m) = 3\alpha(m) - 3\phi(2m)/20;$$

(40.5) $$M_{118}(4n) = M_2(n).$$

Noting that $M_{124}(m)=2M_{244}(m)$, we have,

(41.1) $$M_{124}(m) = 2\beta(m),\ \phi(m)/7, \text{ or } \phi(m)/10,$$

according as $m\equiv 1 \pmod 8$, $\equiv 5 \pmod 8$, or $\equiv 3 \pmod 4$;

(41.2) $$M_{124}(2m) = \alpha(m);$$

(41.3) $$M_{124}(4m) = g\phi(m),$$

where g is defined in (32.2);

(41.4) $$M_{124}(8n) = 12\alpha'(n) + M_4(2n).$$

Now $M_{128}(m)=M_{122}(m)/2$ if $m\equiv 3 \pmod 4$. But if $m\equiv 1 \pmod 4$,

$$\begin{aligned} M_1(m) &= N[m = f_1;\ \text{just three of } x_1, x_2, x_3, x_4 \text{ odd}] \\ &\quad + N[m = f_1;\ \text{just one of } x_1, x_2, x_3, x_4 \text{ odd}] \\ &= 8N[m = f_{224};\ x_1x_2x_3 \text{ odd}] + 4M_{13}(m). \end{aligned}$$

Also

$$M_{122}(m) = M_{188}(m) + 2N[m = f_{224};\ x_1x_2x_3 \text{ odd}].$$

Therefore

$$M_1(m) - 4M_{122}(m) = 4M_{13}(m) - 4M_{128}(m);$$

and

(42.1) $$M_{128}(m) = 3\phi(2m)/40 - \alpha(m)/2,$$

if $m\equiv 3 \pmod 4$;

(42.2) $M_{128}(m) = 3\alpha(m)/2 - \phi(2m)/8,$

if $m\equiv 1 \pmod 4$;

(42.3) $M_{128}(2n) = M_2'(n).$

Since $M_{144}(m)=M_{12}(m)$, if $m\equiv 1 \pmod 4$, we have

(43.1) $M_{144}(m) = \alpha(m) - \phi(2m)/20$ or 0,

according as $m\equiv 1$ or $3 \pmod 4$;

(43.2) $M_{144}(2n) = M_{21}(n)$;

(44.1) $M_{148}(m) = 2M_{244}(m) = 2\beta(m)$ or $\phi(m)/7,$

according as $m\equiv 1$ or $5 \pmod 8$;

(44.2) $M_{148}(m) = 0$ if $m \equiv 3 \pmod 4$;

(44.3) $M_{148}(2m) = M_{12}(m) = \alpha(m) - \phi(2m)/20$;

(44.4) $M_{148}(4n) = M_2(n)$;

(45.1) $M_{188}(m) = M_{128}(m) = 3\alpha(m)/2 - \phi(2m)/8,$

if $m\equiv 1 \pmod 4$;

(45.2) $M_{188}(m) = 0$ if $m \equiv 3 \pmod 4$;

(45.3) $M_{188}(2n) = M_3'(n)$;

(46.1) $M_{222}(m) = M_{112}(m)/3,\ 3M_4(m)/4,\ M_4(m)/2,$ or $M_4(m)/4,$ that is, $\beta(m)$, $3\phi(m)/20$, $3\phi(m)/14$ or $\phi(m)/20$, according as $m \equiv 1$, 3, 5, or $7 \pmod 8$;

(46.2) $M_{222}(2m) = \alpha(m)$;

(46.3) $M_{222}(4m) = g\phi(m),$

where g is defined in (32.2);

(46.4) $M_{222}(8n) = 12\alpha'(n) + M_4(2n)$;

(47.1) $M_{224}(m) = M_{122}(m)/2 = \alpha(m)/2 - \phi(2m)/40,$

if $m\equiv 1 \pmod 4$;

(47.2) $M_{224}(m) = 3\phi(2m)/40 - \alpha(m)/2,$

if $m\equiv 3 \pmod 4$;

(47.3) $M_{224}(2n) = M_{21}(n)$.

Since $M_{228}(m) = M_{244}(m)$ or $2M_{244}(m)$, according as $m \equiv 1$ (mod 4) or 3 (mod 8),

(48.1) $M_{228}(m) = \beta(m)$ or $\phi(m)/14$,

according as $m \equiv 1$ or 5 (mod 8);

(48.2) $M_{228}(m) = \phi(m)/10$ or 0,

according as $m \equiv 3$ or 7 (mod 8);

(48.3) $M_{228}(2m) = M_{12}(m) = \alpha(m) - \phi(2m)/20$;

(48.4) $M_{228}(4n) = M_2(n)$;

(49.1) $M_{248}(m) = M_{128}(m)/2 = 3\alpha(m)/4 - \phi(2m)/16$,

if $m \equiv 1$ (mod 4);

(49.2) $M_{248}(m) = 3\phi(2m)/80 - \alpha(m)/4$,

if $m \equiv 3$ (mod 4);

(49.3) $M_{248}(2n) = M_{22}(n)$.

Since $M_{288}(m) = M_{222}(m)/3$ or $M_{222}(m)$, according as $m \equiv 3$ or 1 (mod 8);

(50.1) $M_{288}(m) = \beta(m)$, $\phi(m)/20$, 0, or 0,

according as $m \equiv 1$, 3, 5, or 7 (mod 8);

(50.2) $M_{288}(2m) = M_{13}(m) = \alpha(m)/2 - \phi(2m)/40$;

(50.3) $M_{288}(4n) = M_4(n)$;

(51.1) $M_{444}(m) = M_{13}(m) = \alpha(m)/2 - \phi(2m)/40$

if $m \equiv 1$ (mod 4);

(51.2) $M_{444}(n) = 0$ if $n \equiv 2$ or 3 (mod 4);

(51.3) $M_{444}(4n) = M_1(n)$.

Since $M_{448}(m) = M_{148}(m)/2$ if $m \equiv 1$ (mod 4), we have

(52.1) $M_{448}(m) = \beta(m)$ or $\phi(m)/14$,

according as $m \equiv 1$ or 5 (mod 8);

(52.2) $M_{448}(n) = 0$ if $n \equiv 2$ or 3 (mod 4);

(52.3) $M_{448}(4n) = M_2(n)$;

(53.1) $M_{488}(m) = M_{248}(m) = 3\alpha(m)/4 - \phi(2m)/16$,

if $m \equiv 1 \pmod 4$;

(53.2) $M_{488}(n) = 0$ if $n \equiv 2$ or $3 \pmod 4$;

(53.3) $M_{488}(4n) = M_3(n)$;

(54.1) $M_{888}(m) = M_{288}(m) = \beta(m)$,

if $m \equiv 1 \pmod 8$;

(54.2) $M_{888}(n) = 0$,

if $n \equiv 2$ or $3 \pmod 4$ or $5 \pmod 8$;

(54.3) $M_{888}(4n) = M_4(n)$.

10. *Some Miscellaneous Results.* Letting f'_{abc} denote the form

$$x_1^2 + ax_2^2 + bx_3^2 + cx_4^2 + 16x_5^2$$

and

$$M'_{abc}(n) = N[f'_{abc} = n],$$

we have

(55) $M'_{114}(8n + 7) = M'_2(8n + 7)/2 = \phi(8n + 7)/20$;

(56.1) $M'_{111}(8n + 7) = 4M'_{114}(8n + 7) = \phi(8n + 7)/5$;

(56.2) $M'_{111}(4m) = 8M_{22}(m) + \lambda(m) = 12a\phi(m)$,

where a is defined in (6);

(56.3) $M'_{111}(8n) = \lambda(2n)$;

(57.1) $M'_{248}(8n + 7) = M_{244}(8n + 7)/2 = \phi(8n + 7)/40$;

(57.2) $M'_{248}(8n + 5) = M_{448}(8n + 5)/2 = \phi(8n + 5)/28$;

(58) $M'_{488}(8n + 5) = M'_{248}(8n + 5) = \phi(8n + 5)/28$.

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