ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society of which the reports have not yet been published. Papers are numbered serially throughout this volume.* Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

385. Professor B. W. Jones: *A new definition of genus for ternary quadratic forms.*

H. J. S. Smith and L. E. Dickson have defined the genus of a ternary quadratic form in terms of the quadratic character, with respect to the prime factors of the Hessian, of the integers represented by the form and its reciprocal form. In this paper it is shown that associated with every form f are certain arithmetic progressions A having the property that, for every integer n, $f \equiv n \pmod{N}$ is solvable for N arbitrary if and only if n does not occur in a progression of A; that is, all the integers represented by the form are contained in progressions not among A. These progressions, together with the Hessian, are proved to determine the genus of f as defined by H. J. S Smith and by L. E Dickson. (Received September 26, 1930.)

386. Dr. W. J. Trjitzinsky (National Research Fellow): *A study of indefinitely differentiable and quasi-analytic functions.*

Among the various developments presented in this paper are some relating to certain consequences of the generalized Cauchy formulas, existing for Borel monogenic functions. These functions form an important class of quasi-analytic functions of a complex variable. For the real domain, representations are given of quasi-analytic functions which contain as special cases certain representations due to Carleman and de la Vallée-Poussin. In particular, necessary and sufficient conditions are found under which a given set of constants will constitute the set of initial values (at a point) of a function of a given quasi-analytic class, the quasi-analytic classes being defined by the law of decrease of the coefficients of certain series. (Received October 2, 1930.)

387. Dr. J. A. Shohat: *On the classical orthogonal polynomials.*

Starting from the linear second-order differential equation satisfied by the classical orthogonal polynomials (Legendre, Jacobi, Hermite, Laguerre)

* See pp. 1–2 and p. 45 of the January issue.

$\phi_n(x)$, and using Darboux's formula, the author shows that $K_n(x) = \sum_{i=0}^{n} \phi_i^2(x)$ satisfies a linear differential equation of the *first order*. This leads, in a very simple way, to some results, believed new, concerning $K_n(x)$, in particular, $K_n\binom{a}{b}$, also $\phi_n{}^1\binom{a}{b}$ and $\int_a^x p(x)\phi_n(x)p_{n-1}(x)dx$, $a < x < b$, where (a, b) is the interval of orthogonality, and $p(x)$ is the corresponding characteristic function. (Received September 25, 1930.)

388. Professor Edward Kasner: *Near-collineations.*

A near-collineation of the plane is defined as a point transformation which converts exactly three simply infinite systems of straight lines into straight lines. If four systems exist, all straight lines are preserved and we have necessarily a collineation (see Kasner, this Bulletin, 1903, p. 46). The author studies certain classes of near-collineations in detail. If two of the systems of lines in both planes are pencils, the third system in both planes is also a pencil. The correspondence between the pencils is not homographic. The following problem is solved: find all near-collineations for which the first system is a pencil converted into a homographic pencil, and the second system in the first plane is a pencil. The analogous theory in space is much more complicated. Near-collineations may preserve $2\infty^3$ or $9\infty^2$ straight lines. (Received October 1, 1930.)

389. Mr. Alfred Korzybski: *On structure.*

After giving the necessary definitions, the author discusses the following topics: similarity of structures; relations as a form of multidimensional order; empirical structures, linguistic structures; the structure of mathematics; the structure of the aristotelian, euclidean, and newtonian systems; the structure of the non-aristotelian, non-euclidean, and non-newtonian systems. Consequences and applications are given. (Received October 2, 1930.)

390. Mr. Alfred Korzybski: *A generalized theory of mathematical types.*

This paper discusses the classical theory of types, introduces structural considerations, and presents the new generalized theory of types. Consequences and applications are given. (Received October 2, 1930.)

391. Mr. Alfred Korzybski: *A non-aristotelian system.*

The author considers a certain non-aristotelian system from the point of view of structure and the generalized theory of types. The "is" of identity on different levels is discussed. An outline of the new system as an applied branch of the study of mathematical foundations is given, with consequences and applications. (Received October 2, 1930.)

392. Professor J. Douglas: *Existence of a surface of absolutely least area bounded by a given contour.*

In his solution of the problem of Plateau, of which the manuscript is in the hands of the editors of the Transactions of this Society, the writer develops

certain formulas relating his fundamental functional $A(g)$ to $S(g)$, the area of the harmonic suface determined by the representation $x_i = g_i(\theta)$ of the given contour Γ, any Jordan curve. In a paper just published (Annals of Mathematics, (2), vol. 31 (1930), pp. 457–469) T. Radó has given an independent solution of the Plateau problem under the restriction that Γ be rectifiable, in which he makes use of a certain theorem of Koebe concerning the possibilty of mapping a polyhedral surface conformally on a circular disc. With the aid of this conformal map, and the above mentioned formulas connecting $A(g)$ with $S(g)$, the writer is able to settle very quickly a question l ft outstanding in both Radó's paper and his own, namely to show that the minimal surface proved to exist has the absolutely least area of any surface bounded by the given contour. In case the Jordan contour is sufficiently crinkly, there is the possibility that this statement may become vacuously true through the circumstance of every area bounded by the given contour being equal to $+\infty$. (Received September 27, 1930.)

393. Professor J. Douglas: *The problem of Plateau for two contours.*

In his paper *A general formulation of the problem of Plateau* (presented to the Society October 26, 1929, abstract in this Bulletin, vol. 36 (1930), p. 50) the author has given methods adequate to solve this problem for any number of contours and any prescribed topology of the minimal surface. In the present paper the work is carried through for two contours (any two non-intersecting Jordan curves) and a minimal surface homeomorphic with a circular ring. The fundamental functional $A(g^{(1)}, g^{(2)})$ involves elliptic functions with periods 2π, $2\,pi$, where p is the logarithm of the ratio of the radii of the ring. A new feature, not present in the one-contour case (topology of a circular disc) is that the existence of the minimal surface depends on whether a certain equation in p has a finite positive real root. For instance, in the classic case of two co-axial circles, radii a and b, distance between centers d (catenoidal surface), this equation is $p^2(a^2+b^2-2ab\cosh p)+d^2\sinh^2 p=0$. The conformal mapping of a plane region bounded by two Jordan curves on a circular ring, including the one-one continuous nature of the map on the boundaries, is contained in this work as a special case. (Received September 27, 1930.)

394. Professor J. Douglas: *The problem of Plateau for a Möbius leaf.*

Here the methods of the paper cited in the previous abstract are applied to solve the problem of the existence of a minimal surface having the topology of a Möbius leaf bounded by a single given contour (Jordan curve). As canonical region on which to represent the minimal surface conformally (with possible singular points) we choose a circular ring, radii 1 and q, where points elliptically inverse with respect to the concentric circle of radius $q^{1/2}$ are regarded as identical. With the use of a two-sided covering surface, the problem amounts to the two-contour case where the two contours are geometrically coincident. The fundamental functional $B(g)$ involves elliptic functions, and the minimal surface depends for its existence on the possibility of a finite positive real root for a certain equation in $p=\log(1/q)$. (Received September 27, 1930.)

395. Professor J. Douglas: *A problem in the topological differential geometry of surfaces.*

According to a formula of Gauss-Bonnet, $\iint K d\sigma = 4\pi(1-p)$, where K is the Gaussian curvature, $d\sigma$ the element of area of a closed surface of genus p (supposed to have continuous curvature, no sharp edges). If $p=1$ (topology of the torus), then $\iint K d\sigma = 0$, which shows that both elliptically and hyperbolically curved regions R_e, R_h must present themselves. In the example of the torus neither R_e nor R_h is shrinkable to a point. The presumption would seem to be that the same is true for every surface of genus 1, but no proof is known. The question relates to the possibility of a solution of the Plateau problem with one contour and of the topology of a torus with a hole in it. If $p>1$, then $\iint K d\sigma < 0$. It follows that the surface cannot be everywhere elliptically curved. Nor can it be everywhere hyperbolically curved, for that is impossible for a closed surface. Thus the same question can be raised in the case of any genus greater than zero. (Received September 27, 1930.)

396. Mr. Hassler Whitney: *A logical expansion in mathematics.*

In a collection of n objects, let $n(A)$ be the number of those with the property A, $n(\overline{A})$ the number not having this property, $n(AB)$ those with both properties A and B, etc. We may write $n(\overline{A}) = n(1-A) = n - n(A)$, $n(\overline{A}\,\overline{B}) = n(1-A)(1-B) = n - n(A) - n(B) + n(AB)$, and in general $n(\overline{A}_1\overline{A}_2 \cdots \overline{A}_k) = n(1-A_1)(1-A_2)\cdots(1-A_k)$. This expansion is useful when the objects with a given property are easier to count than those without. For example, a fundamental formula giving the number of numbers $<x$ not divisible by a set of numbers $p_1, \cdots, p_k$ is exactly this expansion. As another example, in counting the number of ways of coloring a map, we count only those colorings where any pair of adjacent regions are of different colors. If we use the above expansion, and re-group terms, we get the polynomial formula found by Birkhoff. (See the following abstract.) (Received October 4, 1930.)

397. Mr. Hassler Whitney: *A theory of graphs and their coloring.*

The rank of a graph is the number of vertices minus the number of connected pieces. In coloring a graph, any two vertices joined by an edge must be of different colors. Let n_{ij} be the number of sub-graphs of G of rank i which contain j edges. Then the number of ways of coloring G in λ colors is $\sum(-1)^j n_{ij}\lambda^{v-i}$, if G contains v vertices. This result first found (in a different form) by Birkhoff (Annals of Mathematics, vol. 14, p. 42) is proved by a simple logical expansion. (See the preceding abstract.) Recursion formulas are deduced. The numbers n_{ij} for all i, j give a rather complete description of the graph. Suppose a graph G' is in a single piece, and there are no two graphs H_1, H_2 which form G' if they are joined at a single vertex. Then G' is non-separable. Each number n_{ij} for all graphs G is given by a unique polynomial, whose terms are products of numbers of non-separable sub-graphs of different types of G. If we call the linear terms of this polynomial f_{ij}, n_{ij} is given in terms of f_{kl}, $k \leqq i$, $l \leqq j$, by an arithmetical formula. The question of

when a graph is planar is discussed, and results are found. (Received October 4, 1930.)

398. Dr. A. B. Brown: *Critical sets of an arbitrary real analytic function of n variables.*

The results of earlier papers are extended to the case of a function whose critical points may form an arbitrary complex. This condition is satisfied by any real analytic function of n independent variables. Sets of relations are obtained which reduce to those of earlier papers in the cases there considred. The problem has also been solved independently by Morse (in a paper not yet published) by methods, and with results, more in line with the applications, as compared with the geometrical treatment of the present paper. The author owes thanks to Professor Morse for suggesting the investigation of one question which contributed essentially to the results of the paper. (Received October 4, 1930.)

399. Professor Marston Morse: *The critical points of a function of n variables.*

This paper contains among other results the first treatment of the critical points of a real analytic function of n variables without restriction on the nature of the critical points or the function on the boundary. It includes all previous results known to the author as special cases except the results of Whyburn when the critical values are not isolated. It is much broader than the analytic case. The definition of type numbers does not require the critical set to be a complex, or to consist of a finite number of distinct point sets. It is topologically invariant. In the analytic case the definition is reduced to the determination of relative Betti numbers of regions on analytic manifolds bounded by closed analytic manifolds all without singularity. The relations between the type numbers and the appearance and disappearance of non-bounding cycles is taken up. Finally the function is analytically deformed into a nearby function whose critical points are non-degenerate, lie in the neighborhoods of the given critical sets, and have type number sums at least as great as those of the given sets. (Received October 6, 1930.)

400. Dr. Leo Zippin (National Research Fellow): *On a problem concerning the "aleph-null Bein."*

The following problem has been proposed by Karl Menger: Is each point p of a locally connected continuum C, which contains a subcontinuum not locally connected at p, the vertex of an "aleph-null Bein" in C? (An "aleph-null Bein" is a denumerable sequence of arcs converging to a limit arc, the arcs and limit arc having a single point p in common, and in pairs p only.) A locally connected continuum is constructed having as subset the Knaster continuum (which contains no Jordan arc) in which no "aleph-null Bein" exists. It is shown that every point p, as above, is the vertex of an "omega-Bein"; this last, depending on a result due to Mr. Nöbling, was known to him and to Menger. (Received October 6, 1930.)

401. Dr. Lulu Hofmann: *Plane transformations preserving centers of gravity.* II.

This paper is a continuation of one with the same title read before the Society at the New York meeting in February, 1930. To simplify the wording of the theorems, the condition imposed in (a) [(b)] on the transformation [area] is called the g-condition. The main theorem of the present paper then reads as follows. If in (a) a transformation is to obey the g-condition for a complete set of similar areas (that is, all areas of the set are similar and every area similar to one of the set is contained in it), then ϕ and ψ are harmonic, so that, according to the converse of (1) in the earlier paper, the transformation also obeys the g-condition for all circles. In addition, different sets of areas are studied which in (a) determine the same transformations. Such sets are, for example, the set of all ∞^3 squares parallel to a fixed direction, the set of all ∞^4 squares, the set of all areas such that each area has an axis of symmetry and admits a group of rotations of order greater than two. The last theorem in the earlier paper is corrected so as to describe the set of areas last mentioned. (Received October 24, 1930.)

402. Dr. E. J. McShane: *Semi continuity in the calculus of variations, and absolute minima for space problems.*

A theorem is obtained on the existence of the absolute minimum in the isoperimetric problem which includes all of Tonelli's results on the subject. Certain important theorems, including those of Osgood and Lindeberg and several due to Tonelli, are extended to n dimensions with simplification of the proofs. (Received October 25, 1930.)

403. Professor T. R. Hollcroft: *The bitangential curve.*

The bitangential curve of an algebraic surface is the locus of the points of contact of planes which touch the surface in two points. Its order was found by Cayley for a non-singular surface. Basset (*A Treatise on the Geometry of Surfaces*, Cambridge, 1910, pp. 40 and 280) states that he has been unable to determine the reduction in the order of the bitangential curve due to nodal and cuspidal curves of the surface. In the present paper, by combining certain formulas due to Cayley and Zeuthen, there is found the order of the bitangential curve for a surface of given order which has nodal and cuspidal curves of given orders and also multiple points and planes of given orders. (Received October 18, 1930.)

404. Mr. Ainsley H. Diamond: *Quadrilaterals inscribed and circumscribed to a plane cubic.*

The 54 incomplete quadrilaterals circumscribing and inscribing a non-singular cubic may be divided into 9 sets of 6 each, each set corresponding to one of the inflexions. The diagonals of any quadrilateral of a set intersect in the inflexion corresponding to the set. Again the quadrilaterals may be divided into 6 sets of 9 each. The 12 vertices of 3 quadrilaterals appropriately chosen lie by threes on 12 straight lines if chosen from the same set, or by

threes on 4 straight lines if chosen from different sets. The 12 vertices of 3 quadrilaterals of a set which correspond to 3 collinear inflexions from 2 pairs of triangles, each pair being triply in perspective with respect to these inflexions. A nodal cubic is circumscribed and inscribed by 3 quadrilaterals; a cuspidal cubic by none. (Received October 31, 1930.)

405. Dr. J. L. Dorroh: *Some metric properties of descriptive planes.*

A plane satisfying Axioms I–VIII of Velben's *System of axioms for geometry* is called a *descriptive plane. Limit point* is defined in terms of the interiors of triangles. It is shown that a descriptive plane is metric if it contains a countable set of distinct points which has a limit point and that a descriptive plane is in one-to-one continuous correspondence with an everywhere dense subset on the number-plane if it contains a separate segment. Also, it is shown that not every descriptive plane is metric and that not every metric descriptive plane contains a separate segment. (Received October 10, 1930.)

406. Professor Raymond Garver: *Invariantive aspects of a transformation on the Brioschi quintic.*

A number of more or less general transformations on algebraic equations have been devised which lead to transformed equations whose coefficients are invariants (or covariants) of one or more forms. The present paper considers a certain well-known and important transformation on the Brioschi normal quintic, showing that it, or slight modifications of it, can be interpreted as special cases of three such transformations, and discussing the use of this information in the setting up of the transformed equation. (Received October 31, 1930.)

407. Professor E. R. Hedrick: *Theorems associated with Liouville's Theorem for non-analytic functions.*

In a paper presented before this Society on November 29, 1929, the author showed (see this Bulletin, vol. 36, p. 59) that the theorem that the absolute value of a function of a complex variable cannot have a maximum is extensible to the case of functions that are non-analytic, and that the Liouville theorem is also extensible under certain restrictions. In the present paper, the restrictions imposed are removed, and the theorems are presented in very general form, the only condition not present in the usual theorems being that the jacobian shall not vanish. Similar theorems are proved also for three and for n dimensions. (Received October 29, 1930.)

408. Dr. D. H. Lehmer: *A new calculus of numerical functions.*

There are two outstanding theories of numerical functions based on sums of the type $\sum_{n} f(a) \cdot g(b)$, where the summation extends over all solutions (a, b) of $\psi(a, b) = n$. These theories correspond to $\psi(a, b) = a + b$ and ab, and are respectively the calculus of finite differences and the divisor calculus.

In the present paper are discussed the fundamentals of a calculus in which $\psi(a, b) = [a, b]$, the L.C.M. of a and b. The concepts and properties of Dedekind inversion, ideal multiplication (in the sense of Bell), factorable functions ($f(m)f(n) = f(mn)$ for m and n coprime) and r-th divisors have close analogues in this new theory. The fundamental numerical functions such as ϕ, μ, σ, λ, play important roles in this calculus and new identities between them are disclosed. The essential differences between the divisor calculus and the L.C.M. calculus are of such a nature as to render the algebra of the latter very irregular. (Received October 11, 1930.)

409. Dr. Gordon Pall: *Large numbers are sums of four or five values of a quadratic function of x.*

We can suppose that $f(x) = \frac{1}{2}(mx^2+nx)$, where either m and n are odd and relative prime, or $\frac{1}{2}m$ and $\frac{1}{2}n$ are integers of opposite parities and relative prime; and $m>0$. Except when $\frac{1}{2}m$ is odd and $\geqq 5$, every sufficiently large integer is a sum of four values of $f(x)$ for integers x; and, if also $m \neq 6$, this is true for $x \geqq 0$. We obtain an explicit determination of all integers which are sums of five, six, or seven values of $f(x)$, for integers x, and a nearly explicit determination of all which are sums of four values. We find all integers which are not sums of four values of $3x^2+2jx$, ($j=1$ or 2), for integers $x \geqq -k$, where k is given and $\geqq 0$. (Received October 31, 1930.)

410. Professor William M. Whyburn: *Critical sets for functions of n real variables.*

A function $f(x_1, \cdots, x_n)$ with continuous first partial derivatives in a closed and connected portion of real n-space is considered. The critical sets (see this Bulletin, vol. 35, p. 703) of f are grouped into four classes on the basis of the number of complementary domains of these sets which have boundary points in the sets. Conditions are given under which the critical sets are identical with the maximal connected subsets of the collection of critical points of f. In case the number of critical sets of f is finite it is shown that the existence of k minimal sets necessitates the existence of at least $k-1$ sets of intermediate type. The classification is consistent with existing classification of isolated critical points. (Received October 27, 1930.)

411. Professor O. H. Rechard: *The expansion problem associated with a class of irregular ordinary differential boundary value problems.*

This paper investigates the convergence of the expansion of an arbitrary vector in terms of the characteristic solutions of a properly restricted homogeneous vector differential system which is ordinary, linear, and of order n. As is usual in problems of this type, the question of the convergence of the expansion in terms of the characteristic solutions of the system is replaced by the formally equivalent question of the convergence of a sequence of complex contour integrals in the plane of the parameter of the system. The irregularity of the boundary value problem considered arises from the fact that

no restrictions are placed upon the exponents of this parameter in the terms of the characteristic equation of the system. Because of the absence of such restrictions it is in general the case that the convergence of the sequence of integrals cannot be shown by the known methods of procedure. It is shown, however, in any case that the sequence is summable in a defined sense to the arbitrarily given vector. The summation is accomplished by the use of an adaptation of the Riesz typical means. (Received October 1, 1930.)

412. Professor V. G. Grove: *The transformation E of nets.*

In this paper the author extends the notion of the transformation of Ribaucour to nets in ordinary space not necessarily the lines of curvature on the sustaining surfaces. Two nets in relation C are said to be E transforms if an only if every point on the line of intersection of corresponding tangent planes is equidistant from the corresponding points of the sustaining surfaces of the two nets. Several of the properties of the transformation of Ribaucour are also properties of the transformation E. In particular, if a surface S is mapped conformally on a surface $\overline{S}$ by a transformation E of nets on these surfaces, these nets are orthogonal nets in relation K. (Received October 3, 1930.)

413. Dr. H. P. Doole: *An expansion problem for a first order differential equation where the coefficient of y is λx^h, $(h>0)$, $(-a<x<b)$.*

The convergence of the expansion of an arbitrary function $f(x)$ in terms of the solutions of $dy/dx+\lambda x^h y=0$, $(h>0)$, with boundary conditions $y(-a)=y(b)$, $(a\neq b)$, is investigated for all values of h greater than zero. When x^h is positive throughout the interval the expansion converges, when x^h is complex divergence is shown, and when x^h changes sign the expansion converges to $f(x)$ plus additional terms involving $f(-a)$ and $f(b)$. (Received October 13, 1930.)

414. Dr. H. P. Doole: *The determination of the limiting value of the integral used in generalization of a certain lemma.*

The evaluation of the contour integral $\mathrm{Lim}_{m=+\infty}(\frac{1}{2}\pi i)\int_{\Gamma_m} e^{-pz}dz/[1-e^{-2z})z^k]$ is extended to the new cases where k is any positive integer, and where p may have any value. The results are peculiarly interesting in that they contain functions of $[p]$, the largest integer algebraically less than p. This lemma is used in an expansion problem in a paper to follow. (Received October 13, 1930.)

415. Dr. Leo Zippin: *Condition that a locally connected continuum contain an Aleph-Null Bein.*

In answer to a question of Professor Menger's, it is here shown that a necessary and sufficient condition that a locally connected continuum C contain an Aleph-Null Bein, in the sense of Menger, is that there exist in C an arc L, and a set of continua $M_1, M_2, \cdots$ of which L is the limit set, such

that $M_i \cdot M_j = 0$, $i, j = 1, 2, 3, \cdots$ and $i \neq j$, and $M_i \cdot L = 0$ for every i. In this case, every point of L is the vertex of an Aleph-Null Bein of C, and a construction is given. (Received October 14, 1930.)

416. Dr. Ralph P. Agnew: *On range of inconsistency of regular transformations, and allied topics.*

The paper considers into what sequences a sequence of a stated character may be carried by regular sequence-to-sequence transformations. Any given bounded divergent sequence may be carried into any given bounded sequence by a transformation which is not only regular but satisfies also further important conditions. Six theorems are given to show what may be expected of specialized real regular transformations when they are applied to divergent sequences; one of these supplements the *Kernsatz* given by Knopp in the Mathematische Zeitschrift of November 1929. It is also shown that a specialized regular transformation can be found which carries a given unbounded sequence into any given sequence; and that if two sequences converge to the same value, a regular transformation can be found which carries the first into the second. An interesting corollary is the following: Corresponding to each divergent sequence $\{s_n\}$ of complex constants and each complex constant σ there is a regular transformation with a triangular matrix (satisfying important conditions in addition to those necessary to ensure regularity) which evaluates $\{s_n\}$ to σ. (Received October 15, 1930.)

417. Dr. A. A. Albert: *On the Wedderburn norm condition for cyclic algebras.*

In the Transactions of this Society (1914) Professor Wedderburn gave a sufficient condition that a cyclic algebra of order n^2 over a non-modular field F be a division algebra. The condition was $\gamma^r \neq \text{norm } (a)$ for any a in the cyclic field $F(i)$ defining the algebra and any integer r less than n. It has never been shown that this condition is necessary except for the case n a prime. For γ algebras A in sixteen units it seems likely that $\gamma^2 = N(a)$ is possible. The author gives a necessary and sufficient condition on γ that this be so and shows that when the condition is satisfied A is the direct product of two generalized quaternion algebras. Necessary and sufficient conditions that A be a division algebra are given. In particular it is proved that for the interesting case where F is the field of all rational numbers the Wedderburn condition is necessary as well as sufficient. (Received October 17, 1930).

418. Professor R. L. Wilder: *A plane, arcwise connected and connected im kleinen set which is not strongly connected im kleinen.*

In a recent paper G. T. Whyburn (Mathematische Annalen vol. 102, pp. 333–334) raised the question as to whether there exists in the plane an arcwise connected and connected im kleinen set which is not arcwise connected im kleinen. In the present paper there is constructed, in the plane, an arcwise connected set which is uniformly connected im kleinen, but which is not

arcwise connected im kleinen at any point, and, indeed, is not strongly connected im kleinen at any point. (Received October 25, 1930.)

419. Dr. Leo Zippin: *Note on locally cyclicly connected continua.*

A locally cyclically connected continuum C is one in which for every point x and every neighborhood U_x there exists a neighborhood V_x such that every two points of V_x belong to a simple closed curve of C in U_x (compare abstract 36–9–329, G. T. Whyburn). It is readily shown that this condition on C is equivalent to the condition that every point of C is an avoidable point; and the equivalence holds for sets which are connected and connected im-kleinen G-subsets of a locally connected continuum. The purpose of the note is to show that every two points of C form the vertex-set of a "c-spindle" in C, i.e., a set of arcs, uncountably infinite, each arc having the vertex-points as endpoints, and having in pairs no other common point. Moreover if each arc minus its endpoints is regarded as an element, then the set of elements is closed, dense in itself, and totally disconnected. (Received Oct. 29, 1930.)

420. Professor G. E. Wahlin: *The rank equation of an algebra.*

This paper shows that some properties derived by Dickson (*Algebra and their Arithmetics*) for the minimal equation of an element of an algebra can be extended to the rank equation of an albegra A over an infinite field if A has a modulus. (Received October 31, 1930.)

421. J. M. Thomas: *Matrices of integers ordering derivatives.*

Riquier has employed a matrix of integers to arrange the partial derivatives of a set of functions in a more or less determinate order. The matrix establishing a given set of order relations is not uniquely determined. Two matrices defining the same order relations will be called equivalent. The present paper studies the equivalence problem. It gives a method of reducing any such matrix to a canonical form having the property that two matrices are equivalent if and only if identical when reduced to canonical form. All transformations which preserve the order relations established by a given matrix are determined in the sense that every such transformation can be expressed as a product of the transformations described in the paper. Some of these can be applied to all matrices, some to only a restricted class. (Received November 1, 1930.)

422. Professor L. M. Graves: *On the problem of Lagrange.*

For calculus of variations problems in the plane, Tonelli (*Calcolo delle Variazioni*, vol. 2, pp. 89, 318, 486, 557) has derived the equation characterizing a minimizing curve under the very weak hypothesis that the function or functions defining that curve shall satisfy a Lipschitz condition. Using the same restriction on the minimizing functions, a multiplier rule for the problem of Lagrange is obtained, and also an analogue of the Weierstrass condition. The theorems of Hildebrandt and Graves on implicit functions and differential equations in general analysis (Transactions of this Society, vol. 29 (1927),

pp. 127, 514, and especially p. 542) are applied to obtain these results. (Received November 3, 1930.)

423. Professor L. M. Graves: *A transformation of the problem of Lagrange.*

The problem of Lagrange in the calculus of variations, in which admissible curves are required to satisfy certain differential equations, can be transformed into one in which these differential equations are replaced by integral equations. For the transformed problem, a multiplier rule is derived in which the multipliers are all constants. When the inverse transformation is applied to this rule, non-constant multipliers appear, and the well-known form of the multiplier rule is obtained. (Received November 3, 1930.)

424. Miss Grace Shover and Professor C. C. MacDuffee: *Ideal multiplication in a rational linear associative algebra.*

In the extension to rational algebras of the theory of ideals by means of matrices with rational integral elements (MacDuffee, Transactions of this Society, vol. 31, p. 71) a satisfactory treatment of ideal multiplication was lacking. This has now been supplied. If the non-singular left ideals $\mathfrak{a}=(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\mathfrak{b}=(\beta_1, \beta_2, \cdots, \beta_n)$, $\mathfrak{ab}$ are represented by the ideal matrices A, B, C respectively, then C is the greatest common right divisor of the matrices $AS(\beta_i)$, $i=1, 2, \cdots, n$, and also of the matrices $B\overline{R}(\alpha_i)$, where $S(\alpha)$, $\overline{R}(\alpha)$ are the second and transposed first matrices of the number α. In the special case of algebraic fields, a basis in canonical form for the ordinary ideal product is thus readily obtained by rational operations. (Received November 3, 1930.)

425. Professor L. W. Griffiths: *Certain universal functions of generalized polygonal numbers.*

The methods and results in this paper are similar to those in the author's paper *A generalization of the Fermat theorem on polygonal numbers* (Annals of Mathematics, (2) vol. 31 (1930), pp. 1–12.) The summands are the value of $g(x)=x+m(x^2-x)/2$ for $x=0, \pm 1, \pm 2, \ldots$. The sum w of the coefficients $\leqq m-2$. There is no conclusion if $w=m-2$ and the first four coefficients are 1, 1, 2, 3 or 1, 1, 2, 4. Otherwise all universal functions are determined. (Received November 4, 1930.)

426. Miss Marguerite Zeigel: *Principal directions for two dimensional surfaces in hyperspace.*

Several generalizations of principal directions on an ordinary surface have been suggested for two dimensional surfaces in hyperspace. Wilson and Moore (Proceedings of the American Academy of Arts and Sciences vol. 52 (1916) p. 350) after discussing the limitations of a generalization proposed by Kommerell have suggested two other possibilities. This paper suggests another generalization, which in some respects seems more analogous to the corresponding idea for ordinary surfaces. The treatment of principal directions for ordi-

nary surfaces consists in projecting the principal curvature vector upon the normal to the surface, and in taking as principal directions those for which this projection is an extreme. It is proposed to employ the same method for a surface in hyperspace. Since in this case there may be three independent normals to the surface, we select the *mean normal vector* as the one on which to project the principal curvature vector. This normal vector, which is a proper invariant (hence independent of the direction of the curve), is a natural choice. In ordery space, even for minimal surfaces, the proposed principal directions reduce to the usual form. (Received November 4, 1930.)

427. Professor P. R. Rider: *On small samples from certain non-normal universes.*

This paper applies a method previously developed (Biometrika, vol. 21, pp. 124–143) to obtain the distributions of the ratio of mean to standard deviation in small samples from a triangular and from a U-shaped universe. The distributions of means of samples from these two universes are also given. The probability corresponding to an interval of three sample standard deviations on each side of the sample mean is discussed. This statistical study was made possible by a grant-in-aid from the National Research Council for the assistance of a computer. (Received November 4, 1930.)

428. Professor F. L. Wren: *A new theory of parametric problems in the calculus of variations.*

The integral to be minimized is

$$I=\int_{t_1}^{t_2}[2f(x^1, x^2, \cdots, x^n; x^{1\prime}, x^{2\prime}, \cdots, x^{n\prime}]^{1/2}dt=\int_{t_1}^{t_2}[2f(x, x')]^{1/2}dt$$

where $f(x, x')$ is positive, homogeneous of degree two in x', and of class C^{iv} in a certain region of $2n$-dimensional points (x, x'). The first and second variations of the integral I are studied and the necessary conditions for a minimum derived therefrom. In studying the second variation it is found desirable to use orthogonal variations, which are those variations η^i which satisfy the relation $\int_{\alpha\beta}\eta^\alpha x^{\beta_1}=0$. Certain properties of the analogues of the Jacobi equations are discussed and a mechanism for determining conjugate points is set up. The Hamilton-Jacobi theory for the integral I is also discussed. (Received November 5, 1930.)

429. Mr. L. J. Paradiso: *Solution of bounded variation of a functional equation.*

In this paper a functional equation of the type $\phi(x)=f(x)+\lambda\int_a^b K(x, s)d\phi(s)$ is studied, where $K(x, s)$ is of uniformly bounded variation in x for each s in (a, b) and continuous in s for each x in (a, b). In order to develop the theory of this functional equation it is necessary to prove some new theorems involving Stieltjes integrals. The method of successive substitution is applied to obtain the solution $\phi(x)$ of this equation. It is shown that a solution $\phi(x)$ exists whenever the absolute value of λ is less than the reciprocal of the total variation of $K(x, s)$ in x in (a, b) and the solution is displayed in the form $\phi(x)=f(x)-\lambda\int_a^b\Phi(x, s; \lambda)df(s)$ where $\Phi(x, s; \lambda)$ is the resolvent kernel of

$K(x, s)$ defined by an infinite series which is absolutely and uniformly convergent for all x and s in the rectangle $a \leqq x \leqq b$, $a \leqq s \leqq b$. To remove this restriction on λ additional independent hypotheses are imposed on $K(x, s)$. The Fredholm theory of this functional equation is then developed and conditions are obtained for the existence of the integrals involved. (Received November 5, 1930.)

430 Dr. J. H. Roberts: *A continuous curve in 3-space.*

In the author's thesis it is shown that if M is a continuous curve in the plane S, then there is a continuous (1, 1) transformation T of S into itself such that the subset of $T(M)$ within any rational rectangle is the sum of a finite number of connected sets. The present paper gives an example which shows that if in the above statement "the plane" and "rectangle" are respectively replaced by "3-dimensional space" and "cube," then the resulting proposition is not true. It is shown that if M is a continuous curve and G is any uncountable collection of distinct spheres in 3-dimensional space S, then there is at least one sphere R (of radius r) of the set G such that for every positive number ϵ there is a subset K of $R-M$, such that $M-K=s_1+s_2+\cdots+s_n$, where $s_i(i \leqq n)$ is connected, and either lies within the sphere concentric with R and of radius $r+\epsilon$, or lies outside the sphere concentric with R and of radius $r-\epsilon$. (Received November 5, 1930.)

431. Professor H. E. Buchanan: *Small oscillations of the neutral helium atom near the equilateral triangle positions.*

This paper discusses the character of small oscillations of the neutral helium atom, from the standpoint of the classical mechanics, when the masses of the negatively charged electrons are finite. Much new information is thus obtained which is lacking when the masses are considered to be infinitesimal. All the results of Crudelli in his memoir published in the Rendiconti, 1926, are obtained here with less algebra and many new theorems are proved. The characteristic exponents for finite values of the masses are, O, O, $\pm KeVi$, $\pm Ke(\alpha \pm i\beta)$, $\pm KeVi$, $\pm KePi$, where e is the charge on each electron. Thus the only possible periodic oscillations are three in number and their periods are inversely proportional to the charge. (Received November 7, 1930.)

432. Professor Gillie A. Larew: *Discontinuous solutions for the anormal case of the Lagrange Problem in the calculus of variations.*

Caratheodory, (Acta Mathematica, vol. 47) has investigated the Problem of Lagrange in the calculus of variations by the method of geodetic equidistance. This treatment does not require the customary hypothesis ruling out the case called by Hahn "anormal." The difficulty has not, however, been eliminated, since, in the anormal case, the extremals lie on a surface in n-space, and solutions, if they exist, must consist of curves with at least one corner. In this paper it is shown that under proper hypotheses there can be constructed in the neighborhood of any extremal with a corner a family of extremals, each with one corner. This family is shown to form a field in which the usual Weierstrass construction is available. (Received November 7, 1930).