DIFFERENTIAL GEOMETRY IN THE LARGE*

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1. *Introduction.* As a typical example of a theorem in this type of differential geometry we take the so-called *four-vertex theorem.* It states that *every closed plane convex curve has at least four different points of extreme curvature.* Thus the ellipse has exactly four such points. Since they are called *vertices,* we use that name for points of extreme curvature in general.

Curvature is a property of differential geometry. Its existence is established by the application of differential calculus to geometry. In the four-vertex theorem a relation is found between the curvature of a general type of curve at points at finite distance. It is obvious that the differential calculus alone cannot lead to such theorems; we need the integral calculus. Therefore, we might call this type of geometry integral geometry. But it has already become customary to call it *differential geometry in the large* (Differentialgeometrie im Grossen).

2. *Proof of the Four-Vertex Theorem.* We shall now give a proof of the four-vertex theorem, to show the nature of such demonstrations. We take a closed plane convex curve having two, and only two, tangents in every direction. We may use this property as a definition, and call such a curve an *oval.* If, however, we define a plane closed convex curve as a continuous closed curve with no more than two points in common with a straight line, we must investigate the relation between both definitions.† Still other ways can be found to define convexity.

* An address presented at the invitation of the program committee at the meeting of the Society in New York City, April 18, 1930, as part of a symposium on differential geometry.

† See A. Emch, American Journal of Mathematics, vol. 35 (1913), p. 407; S. Nakajima, Tôhoku Mathematical Journal, vol. 29 (1928), p. 227.

Let $\mathbf{r}=\mathbf{r}(s)$ be the vector equation of the oval, $\mathbf{r}$ having the components x and y, while s is the arc length measured from a point on the oval. If $\mathbf{t}$ and $\mathbf{n}$ are the unit vectors in the tangent and normal directions, uniquely defined by the assumption of a sense of direction on the oval and a sense of rotation in the plane, we have

$$\frac{d\mathbf{r}}{ds} = \mathbf{t}; \quad \frac{d\mathbf{t}}{ds} = k\mathbf{n}; \quad \frac{d\mathbf{n}}{ds} = - k\mathbf{t},$$

where $k=1/\rho$ is the curvature. Then

$$\int_{s_1}^{s_2} \mathbf{r}dk = \mathbf{r}k\Big]_{s_1}^{s_2} - \int_{s_1}^{s_2} kd\mathbf{r}$$

$$= \mathbf{r}k\Big]_{s_1}^{s_2} - \int_{s_1}^{s_2} k\mathbf{t}ds = \mathbf{r}k + \mathbf{n}\Big]_{s_1}^{s_2};$$

therefore, if the integral is taken around the curve

$$\int \mathbf{r}dk = 0.$$

Now $k(s)$ has certainly one maximum and one minimum, if the oval is not a circle. If we take the origin at one of the points where $dk=0$, we see immediately from the figure that the assumption that there is only one other point for which $dk=0$ leads to an absurdity. The ellipse has four vertices, therefore every oval has at least four vertices.

Among the theorems closely connected with the four-vertex theorem we mention the following:

*On every oval there are at least three pairs of points with parallel tangents and equal curvatures.**

If an oval has n vertices, a circle has at most n points in common with the oval.

In space we can define an *ovaloid* as a closed continuous surface with two and no more than two tangent planes in every direction. A theorem analogous to the four-vertex theorem has for several years been known as the conjecture of Carathéodory.

* W. Blaschke, Archiv der Mathematik und Physik, vol. 26 (1917), p. 65. G. Szegö, ibid., vol. 28 (1920), p. 183. W. Süss, Jahresbericht der Vereinigung, vol. 33 (1924), Part 2, p. 32; Tôhoku Mathematical Journal, vol. 28 (1927), p. 216. T. Hayashi, Palermo Rendiconti, vol. 50 (1926), p. 96.

It is that on every ovaloid there are at least two umbilical points. The exact formulation and demonstration here is much more complicated than in the plane case, but it seems that the proof has recently been given by Cohn-Vossen, for any closed analytic surface that is uniquely representable on a sphere.*

A generalization of the four-vertex theorem for curves in space is also known.†

3. *Affine Properties.* The four-vertex theorems belong to the group of euclidean motions. This suggests the possibility of similar theorems for other groups, as the affine group, the Möbius group of conformal transformations, the group of analysis situs. The conception of a closed convex curve is affine, but instead of osculating circles we have osculating conic sections. The minimum number of points with stationary osculating conic section (the so-called sextactic points) is six.‡ This is similar to the four-vertex theorem. There is also a theorem suggested by the property that a straight line, that is, a curve of zero curvature, has no more than two points in common with an oval. The curve of zero affine curvature is the parabola. The theorem states that if at all points of an oval the osculating conic section is an ellipse (so-called elliptic points), the oval has at most four points in common with a parabola. Another way to state it is that every five points on an elliptically curved oval lie on an ellipse.§ The osculating ellipse of an elliptically curved oval with maximum area encloses the given oval; that with minimum area lies entirely within the oval.¶

4. *Projective Properties.* To differential geometry in the large of the projective group there belong a series of investigations which were started by Kneser|| in 1888 and which were con-

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* W. Blaschke, Mathematische Zeitschrift, vol. 24 (1926), p. 617; see also his *Differentialgeometrie*, vol. 3, p. 289; W. Süss, Tôhoku Mathematical Journal, vol. 27 (1926), p. 306; St. Cohn-Vossen, Proceedings Bologna, 1928; see H. Hamburger, Sitzungsberichte Berliner Akademie, 1922, p. 258; Mathematische Zeitschrift, vol. 19 (1924), p. 50.

† W. Süss, Tôhoku Mathematical Journal, vol. 29 (1928), p. 359.

‡ W. Blaschke, Leipziger Berichte, vol. 69 (1917), p. 321, and *Differentialgeometrie*, vol. 2, p. 43. The theorem is ascribed to G. Herglotz and J. Radon.

§ P. Böhmer, Mathematische Annalen, vol. 60 (1905), p. 256.

¶ S. Ogiwara, Tôhoku Science Reports, vol. 15 (1926), p. 503.

|| A. Kneser, Mathematische Annalen, vol. 31 (1888), p. 507; vol. 34 (1889), p. 204.

tinued by Juel.* They deal with points of inflexion, osculation planes, and other projective invariants. One of Kneser's simplest results is that a plane arc free from singularities that has no more than two points in common with a straight line, does not send more than two tangents through one point of the plane. Juel remarks that the classification of algebraic curves in curves of second, third, etc., degree allows a generalization; and he studies the properties of plane curves that have no more than three, or no more than four, points in common with a straight line, and analogous curves in space. Such curves, if they are composed of a finite number of convex arcs whose tangents turn continuously, are called *elementary curves*. The maximum number of points of intersection with a line is called their *order*. Among Juel's theorems we find, for example, the following:

The number of points of inflexion of an elementary curve is even or odd, if the order of the curve is even or odd.

An elementary curve of order four consisting of only one branch and without double points and cusps has at least one double tangent.

Through stereographic projection the osculation circles of a plane curve pass into the intersections of the sphere with the osculation planes of the image of the curve. In this way, projective properties may pass into metrical properties. Kneser found the four-vertex theorem in this manner.

5. *The Topological Group.* Several types of differential geometry in the large are known that are invariant under topological transformations. One of the best known is the theorem on the curvatura integra of a closed surface of connectivity p. It states† that the integral of the Gaussian curvature taken over the entire surface depends only on p: $\int R d\sigma = 4\pi(1-p)$. It is one of the applications of Stokes' integral theorem, which, just as Gauss's analogous theorem, may be considered as a property of differential geometry in the large. Extensions of this

* C. Juel, Det danske Videnskabernes Selskabs Skrifter, 1906, p. 297; Jahresbericht der Vereinigung, vol. 16 (1907), p. 196; Mathematische Annalen, vol. 76 (1915), p. 343. At the meeting, the importance of Juel's results was pointed out by S. Lefschetz. See also H. Brunn, Jahresbericht der Vereinigung, vol. 3 (1892–93), p. 84.

† See W. Blaschke, *Differentialgeometrie*, vol. 1, p. 111.

theorem to spaces of constant curvature and of higher dimensions, are known.*

Here belong also the theorems on the behavior in the large of geodesics on surfaces, the Jacobi condition, and Poincaré's theorem that there is at least one closed geodesic on an ovaloid,† it being plausible that there are at least three. On a closed surface of positive curvature a point describing a geodesic must cross any existing closed geodesic an infinite number of times, so that two closed geodesics necessarily intersect.‡ The problem of closed geodesics is of importance in dynamics, where it leads to periodic motions. Birkhoff and Morse have developed methods under which this problem, and that of extremals in variational problems in general, can be studied in more dimensions and on manifolds of different connectivity.

The Jacobi condition for geodesics on a surface, which associates with a given point on a geodesic another point, called the *conjugate* point, gives rise to the theorem that the locus of the points conjugate to a point on an ovaloid has at least four cusps. On an ellipsoid there are exactly four cusps. It is not yet satisfactorily proved.§

In Riemannian manifolds of more than two dimensions hardly anything except local differential geometry has been studied. An exception might be made for a generalization of the theorem that only on surfaces of constant curvature curves of constant geodesic distance to a point (so-called distance circles) and curves of constant geodesic curvature (so-called curvature circles) are identical. Only in manifolds of constant curvature exist closed curvature spheres that can be contracted to a point without losing their property.||

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* H. Hopf, Göttinger Nachrichten, 1925, p. 131; Mathematische Annalen, vol. 95 (1925), p. 340; these papers discuss the problem from the point of view of analysis situs. The subject was discussed from the point of view of differential geometry by D. J. Struik in a paper read to the Harvard colloquium in the Spring of 1930.

† H. Poincaré, Transactions of this Society, vol. 6 (1905), p. 237; see also W. Blaschke, *Differentialgeometrie*, vol. 1, p. 142. See A. Speiser, Vierteljahrsschrift Naturforschenden Gesellschaft Zürich, vol. 66 (1921), p. 28.

‡ Hadamard, Journal de Mathématiques, vol. 3 (1897), p. 331. See also M. Kerner, Mathematische Annalen, vol. 101 (1929), p. 633.

§ Blaschke, *Differentialgeometrie*, vol. 1, p. 160, from Carathéodory.

|| B. Baule, Mathematische Annalen, vol. 83 (1921), p. 286; vol. 84, p. 202.

6. *Circles and Spheres.* The properties of circles and spheres attracted the attention of the geometers perhaps since the time when the Neanderthal man looked up and saw the sky revolving above his head. Many attempts have been made to characterize these figures, either for purely scientific reasons, to find interesting properties, or in an attempt to express their perfection. We give some of those properties here.

(a) *Of all closed plane rectifiable curves of given circumference the circle has maximum area.*

This property has been known from antiquity, and was perhaps suggested by the experience of generals, who found that the number of troops necessary to surround a town was no measure for its extent. The proposition is known as the isoperimetric problem, and has been the starting point for the finding of many inequalities on closed curves.*

(b) *The circle can be turned around in a square and always remain tangent to the four sides.* Curves that have this property are called *curves of constant breadth.*

(c) *A needle of length less than the diameter of the circle can turn through an angle of* 360° *without hitting the circumference.* The problem to find such curves of minimum area is called Kakeya's problem.†

(d) *There exists a point inside the circle for which all chords passing through it are equal. The same point divides all such chords in two equal parts.* This is called the center property. The first property exists for many more curves, for instance the limaçon. We call such a point a distance point. In the same way there exists an area point for closed surfaces. In this case, however, the sphere is uniquely determined by the property that there exists a point such that all plane sections passing through it are equal. The center property, however, is affine, and characterizes ellipses and ellipsoids.‡

(e) *All geodesics on a sphere are closed.* Some other surfaces of this kind are known, both surfaces of revolution, and also others.§

* See W. Blaschke, *Kreis und Kugel*, 1916; and T. Bonnesen, *Les Problèmes des Isopérimètres et des Isépiphanes*, Paris, Gauthier-Villars, 1929. Both books contain bibliographies.

† S. Kakeya, Tôhoku Science Reports, vol. 6 (1917–18), p. 71.

‡ T. Kubota, Tôhoku Science Reports, vol. 3 (1913–14), p. 235.

§ O. Zoll, Mathematische Annalen, vol. 57 (1903), pp. 108–133; G. Darboux, *Leçons sur la Théorie des Surfaces*, vol. 3, p. 4.

(f) *All geodesics on a sphere are plane.*

(g) *The sphere is the only surface of constant non-zero Gaussian curvature which is without singular lines.* This might be expressed by saying that a sphere cannot be deformed without singularities being formed.* For ovaloids an analogous property can be proved, that is, it can be shown that two continuously curved ovaloids, built up by analytical parts, are congruent or symmetrical, if they have the same line element. If, however, a small part of the surface of the ovaloid is taken away, of diameter less than any preassigned number, infinitesimal deformations are possible. An analytic surface of constant negative curvature without singularities is impossible.†

An ovaloid is therefore uniquely determined (but for translations) if the Gaussian curvature is given as a function of the spherical image.‡

(h) *All distance circles are curvature circles.* See §6.

(i) *All geodesics starting from a point pass through another point.* An unfinished attempt to characterize the sphere by this property exists.§

(j) *All circumscribed cylinders are circular cylinders.* This property is characteristic for a sphere.¶ Ovaloids of constant breadth show a generalization.

(k) *The sphere has constant mean curvature.* Among all closed surfaces of unique spherical image, the sphere is characterized by this property.‖

7. *Intensive Study of the Preceding Properties.* We shall study more closely some of these properties. We define the distance of two parallel tangents to an oval as the breadth of this oval at the corresponding points. Then we have for curves

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* H. Liebmann, Münchener Berichte, 1919, p. 267. H. Weyl, Vierteljahrsschrift der Naturforschenden Gesellschaft zu Zürich, p. 40. S. Cohn-Vossen, Göttinger Nachrichten, 1927, p. 125.

† D. Hilbert, Transactions of this Society, vol. 2 (1900), pp. 87–99. For discussion and precision, see L. Bieberbach, Acta Mathematica, vol. 48, p. 319.

‡ H. Minkowski, Gesammelte Abhandlungen, vol. 2 (1911), p. 125.

§ E. B. Christoffel, Werke, vol. 1, p. 162. See S. Nakajima, Tôhoku Mathematical Journal, vol. 28 (1927), p. 266; C. Carathéodory, Abhandlungen aus dem Mathematischen Seminar Hamburg, vol. 4(1926), p. 297.

¶ W. Blaschke, *Differentialgeometrie*, vol. 1, p. 155.

‖ M. Fujiwara, Tôhoku Science Reports, vol. 3 (1913–14), p. 199.

of constant breadth d the following properties, several of which are due to Euler, who called these curves *orbiform curves*.*

(a) *The circumference C of such a curve $=\pi d$.*†

(b) *The line connecting two corresponding points is perpendicular to the tangent and has therefore length d.*

(c) *The sum of the curvatures at corresponding points is constant.*

(d) *The evolute is a curve of zero breadth, that is, a curve of which one and only one tangent has a given direction.* An example is Steiner's hypocycloid.‡ And vice-versa, the involute of a curve of zero breadth is a curve of constant breadth.

There exists a curious relation between the curves of constant breadth and Buffon's needle problem in probability theory. This was, in fact, the reason that Barbier became interested. A needle of length l is thrown upon a table where parallel lines are drawn at distance $a>l$. Then the probability that the needle will hit a line is $2l/(\pi a)$. If we bend the needle into two parts making an arbitrary angle with each other the probability will not change. Reasoning in this way we can see that any needle bent into a closed convex curve of length L will have the chance $L/(\pi a)$ to hit a line as long as the curve can hit at most one line. If, however, the curve hits only one line in one position, and two lines in another, we can obtain a curve of constant breadth with the same probability of hitting. For this we roll the curve along one of the lines and cut off the part of the curve outside of the envelope formed by the different positions of the other parallel line on the plane of the curve.§

For surfaces of constant breadth, defined as ovaloids with their sets of parallel tangent planes at constant distance, we know, besides,¶ that they are surfaces of constant perimeter,

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* Euler, Acta Academiae Petropolitanae pro 1778 (1781). See G. Loria, *Spezielle ebene Kurven*, vol. 1, 1910, p. 376, or Jordan-Fiedler, loc. cit.

† E. Barbier, Journal de Mathématiques, (2), vol. 5 (1860), p. 273. Barbier mentions Puiseux as interested in these curves. Euler mentions properties (b) and (d).

‡ See also W. Blaschke, Mathematische Annalen, vol. 76 (1915), p. 504.

§ This reference to the needle problem was given, in the discussion, by O. Ore, who suggested, following Barbier, generalizations where the curve can roll not between two parallel lines but between more general curves.

¶ H. Minkowski, Gesammelte Abhandlungen, vol. 2, p. 277. The inverse property is also true; see W. Blaschke and G. Hessenberg, Jahresbericht der Vereinigung, vol. 26 (1918), p. 215.

which means that the tangent cylinders have cross sections of constant area perpendicular to the generating lines. Such surfaces do not, as a rule, have an area $=\pi d^2$, as the analogy with the plane might suggest. This was discovered by Barbier, who showed, besides, how we can obtain an infinite number of such surfaces.

A generalization of curves of constant breadth can be obtained by studying curves that can turn around in a given regular convex polygon, always remaining tangent to the sides. All such curves inscribed in a polygon of n sides have the same circumference, and at least $2(n-1)$ vertices.* For space curves of constant breadth see §9.

Curves possessing a distance point were investigated in a study on the geometrical form of leaves.† They can be considered as conchoids with respect to themselves. For a similar reason curves of constant breadth are their own parallel curves.

Among the chords of a closed curve in a certain direction, there are one or more of maximum length if the curve is an oval. At the end points of such extremal chords the tangents are parallel. Comparing all extremals in different directions, we have one or more of extremal length. For such a chord the tangents at the end are perpendicular.‡

Kakeya's problem has different answers for different cases. An oval satisfying its condition is the equilateral triangle.§ If we ask for continuous curves only, a solution is Steiner's hypocycloid. But it can be shown that there are point sets of arbitrarily small Jordan measure, in which a line-segment of given length can turn through 360°. Here there is, of course, no longer continuity.¶

8. *Inequalities.* A great number of properties of ovals are inequalities. The starting point of the investigation was the *isoperimetrical* inequality between the circumference C and

* M. Fujiwara, Tôhoku Science Reports, vol. 4 (1915), p. 44.

† B. Habenicht, *Die analytische Form der Blätter* (Quedlinburg, 1905); *Beiträge zur Mathematischen Begründung einer Morphologie der Blätter* (Berlin, 1905); see G. Loria, loc. cit., p. 369.

‡ Hayashi, Tôhoku Mathematical Journal, vol. 22 (1923), p. 387.

§ J. Pál, Mathematische Annalen, vol. 83 (1921), p. 311; compare W. B. Ford, this Bulletin, vol. 28 (1922), p. 45.

¶ A. S. Besicovitch, Journal of the Mathematico-Physical Society of Perm, vol. 2 (1920); Mathematische Zeitschrift, vol. 27 (1928), p. 312.

the area A of a closed curve: $4\pi A \leqq C^2$, where the sign $=$ only holds for the circle. The analogous property for solid bodies is $36\pi V^2 \leqq A^3$, where V is the volume, A the area, and the sign $=$ only holds for the sphere.*

Another simple inequality for ovals is $\pi D \geqq C$, $\pi\Delta \leqq C$. Here D is the maximum distance between two parallel tangents, sometimes called the diameter. The minimum distance Δ is sometimes called the thickness. The sign $=$ holds for curves of constant breadth.†

Another inequality, valid if the oval has a curvature defined at every point, is

$$2\pi R \geqq C, \quad 2\pi r \leqq C,$$
$$\pi R^2 \geqq A, \quad \pi r^2 \leqq A,$$

where R and r are the maximum and minimum radii of curvature,‡ and

$$\Delta \geqq 2r, \quad D \leqq 2R, \quad 4D^2 \geqq C^2 - 4\pi A. \S$$

There are other inequalities, sharper than those just mentioned.¶

In affine geometry there are inequalities expressing the isoperimetrical property of ellipse and ellipsoid. The principal ones are for ovals:

$$8\pi A \geqq C_1^3,$$

where C_1 is the affine length of the oval

$$C_1 = \int (\dot{r} \times \ddot{r})^{1/3} dt;$$

and for ovaloids:

$$12\pi V \geqq A_1^2,$$

where A_1 is the affine area of the ovaloid

* T. Bonnesen, Mathematische Annalen, vol. 95 (1926), p. 267.

† A. Rosenthal and O. Szasz, Jahresbericht der Vereinigung, vol. 25 (1910). W. Blaschke, Leipziger Berichte, vol. 67 (1915), pp. 290–298.

‡ A. Hurwitz, Annales de l'École Normale, (3), vol. 19 (1902), pp. 357–408. W. Blaschke, *Kreis und Kugel*, 1916.

§ T. Kubota, Tôhoku Science Reports, vol. 13 (1924–25), p. 14.

¶ T. Kubota, Tôhoku Science Reports, vol. 12 (1923–24), p. 45; S. Fukasawa, Tôhoku Mathematical Journal, vol. 26 (1920), p. 27.

$$A_1 = \iint (\mid EG - F^2 \mid) du dv,$$

where $ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2$.*

The derivation of these inequalities can be closely connected with the conception of mixed area and mixed volume. Take, for instance, two ovaloids, O_1, O_2, given by the vectors, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$. Then the vector $\boldsymbol{r} = \lambda \boldsymbol{r}_1 + \mu \boldsymbol{r}_2$, $\lambda \geqq 0$, $\mu \geqq 0$, determines a third ovaloid O_{12}. Its volume $V(\lambda, \mu)$ can be expressed as a cubic form in λ, μ,

$$V(\lambda, \mu) = V_{111}\lambda^3 + 3V_{112}\lambda^2\mu + 3V_{122}\lambda\mu^2 + V_{222}\mu^3.$$

Here V_{111}, V_{222} are the volumes of O_1,, O_2, and the two quantities V_{112}, V_{122} are called the *mixed volumes* of O_1, O_2. Then we have

$$V_{112}{}^2 \geqq V_{111}V_{122},$$

$$V_{122}{}^2 \geqq V_{112}V_{222};$$

and hence

$$V_{112}{}^3 \geqq V_{111}{}^2V_{222},$$

where the sign $=$ only holds if O_1, O_2 are similar and in similar position. If O_2 is the unit sphere, the last equality gives the isoperimetrical property of the sphere.†

9. *Space Curves*. Closed space curves have, thus far, not had the attention that we might expect. Still, some interesting properties have been detected, of which we shall mention a few.

A space curve of constant breadth can be obtained by taking a closed curve whose normal plane at a point P has only one more point Q in common with the curve, and for which PQ is constant. For such curves PQ is also normal at Q; the chords PQ form a one-sided surface. Such a curve lies on a surface of constant breadth.‡

It is also possible to define curves of constant breadth on the sphere,§ or in conformal geometry sets of ∞^1 circles in the plane having constant conformal breadth.¶

* See Blaschke, *Differentialgeometrie*, vol. 2; also R. Zindler, Wiener Berichte, vol. 130 (1929), p. 289.

† See H. Minkowski, *Gesammelte Abhundlungen*, vol. 2, p. 250.

‡ M. Fujiwara, Tôhoku Mathematical Journal, vol. 5 (1914), p. 731; vol. 8 (1915), p. 1; W. Blaschke, Leipziger Berichte, vol. 66 (1914), p. 171.

§ W. Blaschke, Leipziger Berichte, vol. 67 (1917), p. 290.

¶ T. Takasu, Tôhoku Science Reports, vol. 17 (1928), pp. 273, 345.

The spherical image of a closed space curve on the unit sphere is a closed spherical curve. It is intersected by every great circle.* If it has at most one double point, the torsion must change its sign, if it does not vanish identically.†

The total curvature $\int k ds$ of a closed space curve is $\geqq 2\pi$, where the sign $=$ holds only for plane convex curves. The quantity k is the (first) curvature of the curve.

If $k=$ constant, we call the curve a skew circle. If we consider skew circles of $k=1$ that are not closed, the angle α between an arc of length $l \leqq \pi$ and its chord is $\leqq l/2$, the sign $=$ only holding for an arc of the unit circle.‡

10. *Other Theorems on Ovals.* Some curious theorems on ovals deal with the so-called *curvature centroid* (Krümmungsschwerpunkt) of plane curves. It is the point in the plane defined by the radius vector $\int \mathbf{r} d\phi$. As integrals along the curve

$$\int_0 \mathbf{r} d\rho = \int_0 \frac{\mathbf{r} ds}{\rho}$$

it can be defined as the center of gravity of the curve if it were loaded with a mass distribution proportional to the curvature. Then we have the following two theorems.

The point with respect to which the pedal curve of an oval is a minimum is the curvature centroid.§

If instead of looking for the center of gravity of the loaded curve we ask for the moments of inertia, we find a tensor $\int \mathbf{r}\mathbf{r} d\phi$. The two principal axes, determined by this tensor, may be called *curvature-axes.*¶ Some properties of ovals with respect to such axes have been found.

Some other curious theorems on ovals are those dealing with inscribed or circumscribed figures. In an oval at least one square can be inscribed. At least two similar rectangles can

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* K. Löwner, in Polya-Szegö, *Aufgaben und Lehrsätze*, vol. 2, 1925, pp. 165, 391.

† W. Fenschel, Mathematische Annalen, vol. 101 (1929), p. 239.

‡ H. A. Schwarz, in Blaschke, *Differentialgeometrie*, vol. 1, p. 47. Here occur more theorems on these circles.

§ J. Steiner, Gesammelte Werke, vol. 2, p. 99.

¶ B. Su, Tôhoku Science Reports, vol. 17 (1928), p. 35, calls *curvature axis* the axis with respect to which the moment of inertia is a minimum.

be inscribed.* The same holds for circumscribed figures.† About an ovaloid a single infinity of cubes can be circumscribed, the points of contact forming continuous curves. In the case of an ellipsoid these cubes are congruent.‡ A certain similarity to these theorems is shown by the following theorem, which holds for any closed rectifiable curve.

There exists always at least one set of four points on the circumference of a closed curve that lie on a circle and divide the circumference in four equal parts.

There are many generalizations of this theorem. For a space curve, for instance, we can prove that a similar property holds for four points in a plane.§ Finally, if two congruent closed plane curves without multiple points coincide with each other, when they have three points in common, they must be circles.¶

11. *History of the Subject.* The history of this subject leads from Zenodor via some occasional work of Euler and Steiner to the end of the nineteenth century. The first important work on convex figures was by Brunn.|| Then came Minkowski's monumental work.** An interesting little book on plane convex curves was written by Jordan and Fiedler.†† Minkowski's work has been continued by Blaschke, who showed the beauty of differential geometry in the large in many papers and in three books.‡‡ Under his influence not only many German authors wrote on such subjects, but also the present school of Japanese geometers.

The importance of differential geometry in the large was first, so to speak, officially recognized in this country in an

* A. Emch, American Journal of Mathematics, vol. 36 (1913), p. 407; this Bulletin, vol. 20 (1913), p. 27.

† S. Kakeya, Tôhoku Mathematical Journal, vol. 9 (1916), p. 163.

‡ T. Hayashi, Tôhoku Science Reports, vol. 3 (1913–14), p. 15.

§ Communication to the author by A. Kawaguchi.

¶ T. Kajima, Tôhoku Mathematical Journal, vol. 21 (1922), p. 15; T. Kubota, ibid., p. 21.

|| Brunn, Dissertation, München, 1887; Münchener Berichte, 1894, p. 102.

** H. Minkowski, Gesammelte Abhandlungen, vol. 2, p. 131.

†† Jordan-Fiedler, *Contributions à l'Étude des Courbes Convexes Fermées*, Paris, Hermann, 1912.

‡‡ W. Blaschke, *Kreis und Kugel*, 1916; *Differentialgeometrie*, vol. 1, and vol. 2. These books contain many references to the literature.

address by Kasner to this Society.* Later came two papers by Emch. A report on the subject has been published by Ball.† The most important contributions to this part of geometry have resulted, in this country, from the work of Birkhoff and Morse. Their starting point, however, is not the same as ours; it is the application of the calculus of variations to dynamics.

12. *Conclusion.* The results so far obtained in differential geometry in the large are not very systematic. The demonstrations often carry an ad hoc character. But they have a striking simplicity which is both charming and stimulating. Their way of discovery, their demonstrations, their general character belong to geometry in its most genuine form. It is certainly not true that the rococo time alone has had the privilege of elegance in its mathematical discoveries. It is not true either that the end of the romantic age of the Holy Alliance has witnessed the end of geometry in its best sense. The work of Minkowski, Blaschke and others is not the feeble work of epigones. If we consider also the splendid development of local differential geometry since the discovery of pseudospherical displacement, it certainly seems as though we were not a hundred years from a heroic age of geometry,but rather just in the midst of another one.

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* E. Kasner, this Bulletin, vol. 11 (1904–05), p. 283.

† N. H. Ball, American Mathematical Monthly, vol. 27 (1930), p. 348.