KELLOGG ON POTENTIAL

Foundations of Potential Theory. By O. D. Kellogg. Berlin, Springer, 1929. ix+384 pp.

The book before us is modern in its field of interest, and is nevertheless based entirely on the kind of mathematics with which the advanced undergraduate, or young graduate, is familiar; hence the American student has now an opportunity, if he will substitute a European for an American habit, and buy a book, to be well grounded in the fundamental subject of potential theory. He will thus possess knowledge which not only has contact with much of the past development of the theory of partial differential equations and its applications to "classical" mathematical physics and is most important in its relations to functions of a complex variable, but also continues to introduce him to the ever new in mathematical problems. Suffice it here to mention in the last direction the problem of Plateau, which lately, by the acquisition of new existence theorems, has opened up large domains for investigation.

The range of subject is suggested by the various chapter headings: The force of gravity; Fields of force; The potential; The divergence theorem; Properties of Newtonian potentials at points of free space; Properties of Newtonian potentials at points occupied by masses; Potentials as solutions of Laplace's equation, electrostatics; Harmonic functions; Electric images, Green's function; Sequences of harmonic functions; Fundamental existence theorems; The logarithmic potential. Velocity fields and the equation of continuity are treated with fields of force, the Heine-Borel theorem occurs in connection with the divergence theorem, developments valid at great distances are made essential for the properties of potentials, ellipsoidal coordinates and the potential of the homogeneous ellipsoid are given in connection with potentials as solutions of Laplace's equation (see Jeans's ingenious generalization of such formulas in the problem of equilibrium of rotating fluids!), conformal mapping is treated under the logarithmic potential, and so on. The final section is devoted to the explicit formulas required for the mapping of polygons.

In a subject of this character it cannot be hoped to make a complete exposition. Beyond the fundamentals, the author will limit himself somewhat narrowly, knowing that although he must omit direct treatment of many subjects he will cross them in the line of march in such a way that the student may return to them later. Thus in the chapter on *Sequences of harmonic functions*, Professor Kellogg is able to devote an incidental four pages to development in spherical harmonics. The main line of march here, indeed, throughout the last third of the book, is towards the exploration of harmonic functions defined in arbitrary regions and the Dirichlet problem. This problem, in which the boundary values are assigned continuously, is a central problem for potential theory.

In order to make sound the treatment of integrals over regions and surfaces it is necessary to discuss the nature of the regions and their bounding surfaces with considerable care and detail. Green's and Stokes' theorems are established rigorously in terms of the kind of mathematics possessed by the senior college student, and accordingly this student will have a definite tool for future use. We examine then briefly the set of definitions on which this structure is based (see pp. 112, 113), quoting the essential phrases:

A *regular region of space* is a bounded closed region whose boundary is a closed regular surface. A *regular surface* is a set of points consisting of a finite number of regular surface elements, with the following connections:

(a) Two may have in common either a single point (vertex) or a single regular arc (edge), but no other points;

(b) Three or more may have at most vertices in common;

(c) Any two are the first and last of a chain, such that each has an edge in common with the next;

(d) All those having a vertex in common form a chain such that each has an edge, terminating in that vertex, in common with the next; the last may, or may not, have an edge in common with the first.

Edge, vertex, face and closed surface are then defined. A *regular surface element* (p. 105) is a set of points which, for some orientation of the axes, admits a representation

$$z = f(x,y), \qquad (x,y) \text{ in } R,$$

where R is a regular region of the (x, y) plane, and where $f(x, y)$ is continuously differentiable in R. This is enough of the chain of definitions to indicate the character of the proofs. Obviously the division of the surface into regular elements is not unique, and sometimes regular elements must be subdivided in order to satisfy the requirements of the definition of regular surface.

For the examination of the limiting values of potentials and their derivatives, on approach to surfaces on which distributions lie, there is added the restriction on $f(x, y)$ that it shall have continuous partial derivatives of the second order in R. "Tangential" derivatives of potentials of a single layer and potentials of a double layer are related situations. But whereas a restriction (Hölder condition) on the density of a single layer is necessary for the investigation of the tangential derivatives, the corresponding condition is satisfied automatically in the investigation of the limiting values of the potential of a double layer. Hence the author is enabled to approach the traditional boundary value problems with the mere hypothesis of continuity for the densities of the corresponding distributions.

After an introduction which indicates the ideas underlying several of the older treatments, a systematic analysis of these boundary value problems is carried out in terms of integral equations, which are themselves discussed adequately, for bounded, and for certain classes of infinite, kernels. We have thus solutions of the Dirichlet problem, the Neumann problem, and that where $\partial V/\partial n + h(p) V$, $h(p) \geqq 0$, $\not\equiv 0$, is given on the boundary, for regular regions subject to the mentioned condition on $f(x, y)$. We remark parenthetically that there is a considerable difference between this last problem, solvable in terms of a potential of a single layer, and that where are given boundary values of a quantity $h_1(p)\partial V/\partial n + h_2(p) V$, where $h_1(p)$ may vanish but not identically.

The density distributions on the exteriors of conductors, the exterior Dirichlet problem, and the interior Neumann problem are associated with a characteristic value of the parameter of the integral equation.

The restrictions on the boundary are now dropped, and the next twenty pages (315–338) are devoted to a consideration of the Dirichlet problem for general regions, showing how a solution may be associated with an arbitrary set of values given continuously on the frontier, and establishing criteria for boundary points of the region, dependent on the region merely, at which this harmonic function takes on the assigned boundary value. These criteria are given in terms of barrier functions (Lebesgue) and capacity (Wiener). "Given a domain T, and a boundary point q, the function $V(P, q)$ is said to be a barrier for T at the boundary point q if it is continuous and superharmonic in T, if it approaches 0 at q, and if outside of any sphere about q, it has in T a positive lower bound." "A necessary and sufficient condition that the Dirichlet problem for T, and arbitrarily assigned continuous boundary values, is possible, is that a barrier for T exist at every boundary point of T." Boundary points which admit of barriers are called *regular* boundary points. The solution which has been developed by the method of sequences to correspond to the given set of continuous boundary values takes on these boundary values continuously at q if q is regular; hence the importance of the question as to whether or not a boundary point is regular. From the concept of barrier follow immediately several simple sufficient conditions for this approach, established by Poincaré and Zaremba. Wiener's criterion is fully developed.

The author mentions the limitation that "in all questions of uniqueness the hypothesis on the harmonic function that it be bounded is apt to play an essential part," and gives, for contrast, the function $U=x$ in the three-dimensional domain for which $x>0$, as a harmonic function which is not identically zero but takes on the boundary values zero continuously. However, if, at his suggestion, we make an inversion and Kelvin transformation, in order to obtain an example for a bounded region, the illustration does not remain quite the same. In fact, in the unit sphere, the corresponding function is the familiar fraction in the integrand of the Poisson integral, and fails to approach zero at one point on the surface. A similar feature occurs in the plane.

An investigation of discontinuous boundary value problems would throw a good deal of light on such situations. If a function is harmonic and bounded, say in the unit circle $r<1$, it is determined by boundary values assigned *almost everywhere* on the circumference, even if approach is merely along radii. But if the function is not bounded, and outside a certain class, the situation is different. Thus the function

$$U = \frac{\partial}{\partial\theta}\,\frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

approaches zero along *every* radius, as r approaches 1. It takes on continuously the value zero at every boundary point but one. Even more startling examples may be given. Moreover, in the opinion of the reviewer, discontinuous boundary value problems which refer to distributions of positive and negative mass, finite in total amount, are as important from the point of view of physics as the continuous ones. But if the discussion of such problems is omitted from the book before us, except for bibliographical references, the reason is obvious and sufficient. It requires a different preparation in analysis from the one which is assumed.

The same remark applies to development of arbitrary functions in terms of various kinds of sets of orthogonal functions. Many facts are given; but many others, as interesting and as important, depend upon a different sort of analysis. The author must have turned the pages of Fatou's famous memoir with regret at the exclusion of what has been a touchstone for modern mathematics.

Enough has been said to show the reader the interesting and accessible character of the book before us. The author is known among mathematicians for his skillful and clear expositions, as well as for his original contributions and scholarship. The exposition here is reinforced by a number of important exercises, some of them given for the sake of review and drill, but many also in order to present interesting facts and illustrations. The work is unusually rich in scholarly bibliographical data.

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VOGEL ON EGYPTIAN ARITHMETIC

Die Grundlagen der Aegyptischen Arithmetik in ihrem Zusammenhang mit der 2:n-Tabelle des Papyrus Rhind. By Dr. Kurt Vogel. Munich, Beckstein, 1929. 211 pp.

This book, an inaugural dissertation of the University of Munich, is an important contribution to the history of mathematics. It is of especial interest to American scholars because its appearance is almost simultaneous with that of Chancellor Chace's monumental edition of the Rhind Papyrus, published by the Mathematical Association of America.*

Dr. Vogel's book is divided into two main divisions, first, a general discussion of ancient Egyptian arithmetic (pp. 5–53) and second, the $2:n$ table of the Rhind Papyrus (pp. 53–181). These parts are preceded by a four-page introduction, discussing the place of the Rhind Papyrus in the history of mathematics; and are followed by a resumé of the chief results obtained, pp. 181–195. There are good indexes and a bibliography of 106 titles.

The first part gives a very clear and satisfactory account of Egyptian arithmetic with particular attention to the use of fractions, including an explanation of the technical expressions used in the Rhind Papyrus and also a brief description of ancient Egyptian weights and measures. The methods of the Egyptian calculator are illustrated by frequent examples from the Rhind Papyrus. It is made clear, among other things, that we are justified in crediting him with considerable skill in mental arithmetic, as well as with a thorough grasp of the fundamental processes, including division by a fraction. Finally, Vogel is able to give a definite answer to the question, Did the ancient Egyp-

* Reviewed by D. E. Smith in this Bulletin, vol. 36 (1930), pp. 166–170.