GRAUSTEIN ON GEOMETRY

Introduction to Higher Geometry. By William C. Graustein. New York, The Macmillan Company, 1930. xv+486 pp., 126 figures.

During the last few years a number of excellent texts have appeared in English on analytic geometry of space, all of which either presupposed or treated in a brief and sketchy manner the foundations on which the theory was developed. And in the same time a number of synthetic projective geometries have been issued which adequately cover the field from that point of view.

It is therefore with particular pleasure that geometers welcome the volume under review. While somewhat belated as compared with the other books mentioned, it can now enjoy the double role of both cause and effect in the present re-awakening of interest in algebraic geometry, a field which requires a wider application of various mathematical disciplines than any other for its proper introduction.

For a book of this nature, three questions at once arise: what is its scope, what are its presupposed premises, and what are its methods? The first and second can be answered in a word, while the third question will be considered more in detail. The scope is to fix the foundations for the study of analytic projective geometry, by establishing the coordinate systems in points, lines, and planes, and applying them to an extensive study of entities defined by linear and by quadratic equations, together with a few equations of higher degree. The field is sometimes the real continuum, sometimes complex, the distinction between the two being sharply drawn as occasion arises.

A knowledge of cartesian analytic geometry of two and three dimensions, of elementary properties of determinants and of the theory of equations, and of the processes of the calculus are presupposed, in so far as these subjects are developed in a first college course in each. In fact a very elementary knowledge of determinants is sufficient.

The book begins with a review of determinants and the solution of a system of simultaneous linear equations, bringing in the ideas of a matrix and of linear dependence from a purely algebraic point of view. Then a synthetic definition of projection, of vanishing points and lines, the theorem of perspective triangles and the principle of duality. Homogeneous coordinates are introduced as ratios, first for the plane, from cartesian coordinates, and the line at infinity is carefully adjoined to the euclidean plane. The geometric meaning of linear dependence is now introduced, and an analytic proof of the plane perspective triangle theorem is given. Harmonic section is approached from the metric side, interpreted in terms of positive and negative line segments on a straight line. Line coordinates are defined as the coefficients in the equation of a straight line, and the earlier definition of duality is justified by the identical algebra in point and line coordinates. Cross ratio is introduced metrically, but is at once shown to be invariant. Linear transformations are approached by rigid motions, which are shown to form a group; then follows a systematic treatment of one-dimensional linear transformations and a less extensive one in two dimensions.

Geometry in the complex plane is prettily introduced by means of a figure, slightly distorted, but with consistent algebraic steps based on contradictory premises, which leads to a paradox, a wise pedagogy. The one-dimensional geometry is now continued to include the fixed points, the characteristic, and the concept of involution, with metric applications.

The next two chapters, the finest in the book, now extend the purely projective treatment of linear transformations to two dimensions, including duality, and a sharp distinction between the two subgroups, that of affine transformations, and that of motion.

Conics are defined as loci of points which satisfy a quadratic equation, mostly with real coefficients, and the form depends on the rank of the relatively invariant discriminant. When the first member of the equation is composite, the conic is called degenerate; in the following chapter, a curve defined by the same property is called reducible, while a little later a conic having a double point is called singular, but the first word is most frequently employed. In view of the many applications of conics having a vanishing discriminant, the word degenerate seems particularly unfortunate. The chapters on tangents employ the method of approach along the curve; continuity is assumed throughout. Now follow the usual properties of conics, including the theorems of Pascal and of Brianchon, largely treated synthetically, then projectivity and involution on a conic. The chapter on pencils of conics does not introduce elementary divisors, but does discuss all the possible forms in which two conics may intersect. Poles and polars are brought in rather late, and applied to centers and diameters and foci.

The general theorem that two superposed quadratic involutions on a one-dimensional rational carrier have one pair of common conjugate elements is not mentioned, but it is used repeatedly and a particular proof given in each case.

The part on plane geometry closes with an extensive geometry of the circle; it includes coaxial systems and inversion, with one paragraph on general inversion as to a conic.

The chapter on geometry of three dimensions can now proceed much more rapidly. It includes the ideas of point and plane coordinates, developed without metric ideas, and an outline of the quadric surfaces, their reguli and tangent planes. Stereographic projection of the sphere is shown to belong to the geometry of (metric) inversion. No mention is made of linear systems of quadrics, nor of curves on a quadric. A short introduction to tetracyclic coordinates and to line geometry show the possibility of other interpretations to systems of coordinates.

The book is provided with a bibliography and with an index. In every chapter are lists of examples, which contribute materially to keeping the book lively and interesting. Solutions of the problems are not provided.

The press work and proofreading have been done so well that the book presents a very attractive appearance, and is strikingly free from typographical errors. The work seems a little long, but it would be an embarrassing task to show just what parts should have been omitted.

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