ON A TYPE OF ILLUSORY THEOREM CONCERNING HIGHER INDETERMINATE EQUATIONS*

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1. *Introduction.* Let S be a system of equations, some or all of which need not be algebraic, finite or infinite in number, in a finite number μ of independent variables $x_1, \cdots, x_\mu$. If the total number of sets of positive (greater than zero) integral solutions $x_1 = x_1', \cdots, x_\mu = x_\mu'$ of S is finite, we shall refer to these sets as all the positive integral solutions of S. If S has an infinity of sets of positive integral solutions, we shall assume that a set of conditions C may be imposed on $x_1, \cdots, x_\mu$ such that S subject to C has only a finite number of sets of positive integral solutions, and we shall refer to these as all the positive integral solutions of S (so that reference to C may be omitted). To indicate the variables when it becomes necessary, we shall write $S \equiv S(x_1, \cdots, x_\mu)$.

Considering the generality of S, one would suspect that it is difficult to state any non-trivial theorem about all the positive integral solutions of S. Nevertheless several theorems concerning these solutions (the immaterial restriction to a finite number of equations being assumed) occur in the literature and have the appearance at first sight of being genuine theorems. The proof of the theorem in §2 reveals its true character; the generalization of this theorem in §3 scarcely needs a proof, while the further generalization in §4, devised for the occasion, betrays the nature of all such theorems. It is to be noticed that the theorems are much worse than trivialities; they direct us to prolix and unnecessary tentative calculations to settle questions whose answers are presupposed from the beginning.

2. *An Illusory Theorem.*† *If in all the positive integral solutions* $x_1, \cdots, x_\mu$ *of* $S(x_1, \cdots, x_\mu)$ *we set* $x_i = d_i\delta_i$, (d_i, δ_i *positive integers*) *in all possible ways, then the number of positive integral solutions* $y_1, \cdots, y_\mu$ *of* $S(y_1^2, \cdots, y_\mu^2)$ *is* $\Sigma\lambda(d_1 d_2 \cdots d_\mu)$,

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† Liouville, Journal de Mathématiques, (2), vol. 4 (1859), pp. 271–272; stated without proof; reported in Dickson's *History*, vol. 2, p. 700.

where $\lambda(x) = +1$ *or* -1 *according as the positive integer* x *is a product of an even or an odd number of primes (equal or distinct).*

If the proof had been stated, it would have been seen at once that the theorem is merely an unnecessarily complicated form of the following.

To find the number of positive integral solutions $y_1, \cdots, y_\mu$ *of* $S(y_1^2, \cdots, y_\mu^2)$, *find all the positive integral solutions* $x_i, \cdots, x_\mu$ *of* $S(x_1, \cdots, x_\mu)$, *and actually count those in which each of* $x_1, \cdots, x_\mu$ *is a square.*

If it is not obvious that the method stated in the theorem for finding the required number of solutions is worse than its trivial equivalent, consider a very large x_i, and compare the difficulty of resolving it in all possible ways into the required $d_i\delta_i$ with that of deciding whether or not it is a square by extracting its square root. Both methods presuppose the complete solution of $S(x_1, \cdots, x_\mu)$ in positive integers. The first demands the resolution of all $x_1, \cdots, x_\mu$ in all the solutions into their prime factors, and this is likely to be tentative if only one of the x_i has a large prime factor. In the second, only the extraction of square roots is required, and this is an elementary non-tentative process.

The proof is contained in the mere statement of two properties of $\lambda(x)$: if x, y are positive integers, $\lambda(xy) = \lambda(x)\lambda(y)$, and $\Sigma\lambda(d)$, where the sum extends over all divisors d of x, is equal to 1 or 0 according as x is or is not a square.

Thus the introduction of λ into the problem of finding the number of positive integral solutions of $S(y_1^2, \cdots, y_\mu^2)$ merely gives a uselessly complicated way of recognizing a square. If the theorem states anything that was not already known, it is only the two properties of λ recalled above, both of which are extraneous to the problem.

3. *A Generalization* of the Theorem in* §2. The theorem in §2 can be "generalized" in an endless variety of ways. Thus, consider a positive integral solution $x_1^0, \cdots, x_\mu^0$ of $S(x_1, \cdots, x_\mu)$, and any divisor δ_i^0 of x_i^0. Call the product $\delta_1^0 \cdots \delta_\mu^0$ a divisor product belonging to the solution $x_1^0, \cdots, x_\mu^0$. Then we have the following theorem.

* Gegenbauer, Wiener Sitzungsberichte (Math.), vol. 95, II (1887), pp. 606–609. The above statement is from Dickson's summary, loc. cit., p. 701.

Let $\chi(x)$ be a function for which $\chi(xy)=\chi(x)\chi(y)$ for all values of x, y satisfying a definite condition. Let $X(n)=\Sigma\chi(d)$, where d ranges over all divisors of the positive integer n. Then

$$\sum X(x_1^0)\cdots X(x_\mu^0) = \sum\chi(\delta_1^0\cdots\delta_\mu^0),$$

where on the left the summation extends over all those solutions $x_1^0, \cdots, x_\mu^0$ which satisfy the condition mentioned, while on the right the summation extends over all the divisor-products belonging to these solutions.

It is obvious that the conclusion of this theorem is merely a repetition of the definitions of χ, X. The special case $\chi(n) \equiv\lambda(n)$ is the theorem of §2.

4. *Further Generalizations.* Let $\chi_i(x)$ be a function which takes a definite value when x satisfies a definite condition C_i, and define $X_i(n)$ by $X_i(n)=\Sigma\chi_i(d)$, the notation being as in §3. Then, in the same notation,

$$\sum X_1(x_1^0)\cdots X_\mu(x_\mu^0) = \sum\chi_1(\delta_1^0)\cdots\chi_\mu(\delta_\mu^0),$$

where the summation on the right side refers now to all sets $(\delta_1^0, \cdots, \delta_\mu^0)$ formed from the divisor products $\delta_1^0 \cdots \delta_\mu^0$. The special case $(\chi_i n)=\chi(n)$, $(i=1, \cdots, \mu)$, χ as in §3, is the theorem in §3. This is only a tautology on the definitions of χ_i, X_i.

Some of the special cases of this triviality are less easily seen through than the general theorem, for example, the following generalization of §2. Let c be a constant positive integer, and n an arbitrary positive integer. Define $\lambda(n, c)$ by $\lambda(1, c)=\lambda(n, 1) =1$, and for $c>1$, $\lambda(n, c)=0$ if any prime divides n to exactly a $(jc+h)$th power, $j\geqq 0$, $1<h<c$; $\lambda(n, c)=(-1)^t$ in the contrary case, where t is the number of distinct primes dividing n to exactly a $(jc+1)$th power, $j\geqq 0$, $t\geqq 0$.

Write $\Lambda(n, c)=\Sigma\lambda(d, c)$, the sum extending to all divisors d of n. Then we have the following theorem.

If $c_1, \cdots, c_\mu$ are positive integers, the number of positive integral solutions of $S(y_1^{c_1}, \cdots, y_\mu^{c_\mu})$ is $\Sigma\Lambda(x_1, c_1)\cdots\Lambda(x_\mu, c_\mu)$, the sum extending to all positive integrals solution of $S(x, \cdots, x_\mu)$.

For $c_1= \cdots =c_\mu=2$, this is the theorem of §2 and, like it, is illusory.

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