

LINEAR q -DIFFERENCE EQUATIONS*

BY C. R. ADAMS

1. *Introduction.* Recent years have seen the publication by Nörlund,† Carmichael,‡ and Pincherle§ of several papers whose aim is to describe the state of development attained by the difference calculus and the theory of difference equations. Scant attention, however, has been paid to the subject of q -difference equations, which not only is closely kin to, but may properly be regarded as a part of, the field of difference equations. It may therefore be of some interest to present here a brief summary of what has been done in the theory of linear q -difference equations, together with indications of certain extensions.

The linear difference equation

$$(1) \quad \sum_{i=0}^n a_i(x)f(x + n - i) = b(x)$$

is a particular case of a functional equation which is linear in the unknown function f , and in which the argument of f is repeatedly subjected to the non-singular linear fractional substitution $\{x, (ax+b)/(cx+d)\}$. This more general type of functional equation, however, can always be reduced by means of a linear fractional transformation of the independent variable x to one of two normal forms, the first being (1) and the second

$$(2) \quad \sum_{i=0}^n a_i(x)f(q^{n-i}x) = b(x).$$

* An address presented to the Society at the request of the program committee, February 28, 1931.

† Nörlund, *Sur l'état actuel de la théorie des équations aux différences finies*, Bulletin des Sciences Mathématiques, (2), vol. 44 (1920), pp. 174–192, 200–220; *Neuere Untersuchungen über Differenzengleichungen*, Encyklopädie der Mathematischen Wissenschaften, vol. II C 7 (1923), pp. 675–721. See also *Vorlesungen über Differenzenrechnung*, 1924, 9+551 pp.

‡ Carmichael, *The present state of the difference calculus and the prospect for the future*, American Mathematical Monthly, vol. 31 (1924), pp. 169–183.

§ Pincherle, *Il calcolo delle differenze finite*, Bollettino dell' Unione Matematica Italiana, vol. 5 (1926), pp. 233–242, and Atti della Società per il Progresso delle Scienze, vol. 15 (1927), pp. 153–162.

Essentially this fact was proved by Pincherle* in 1880, although this precise formulation of the statement appears to have been first made by Stridsberg† in 1910. The normal form is of the type (1) or of the type (2) according as the double points of the substitution $x' = (ax+b)/(cx+d)$ are coincident or distinct. Hence it may be said that in general the functional equation of more general type is reducible by a linear fractional transformation to the normal form (2), while in particular it may be so reducible to the normal form (1).

The equation (2) is commonly termed a linear q -difference equation,‡ and its close relationship to the equation (1) is manifest. Indeed it is apparent that by the transformation $x = q^x$ the equation (2) is carried over into an equation of type (1); consequently if solutions can be found for the equation (2) when the coefficient functions $a_i(x)$ and $b(x)$ are of a certain class of functions of x , the problem of obtaining solutions of (1) when the coefficients belong to the same class of functions of the argument q^x is solved. Likewise the inverse transformation carries the equation (1) into one of type (2).

The equation (2) is said to be of *order n*, homogeneous or non-homogeneous according as $b(x)$ is or is not identically zero; for brevity we shall denote the respective cases by (2h) and (2n). It is clear that the substitution $q' = 1/q$ immediately reduces either of the cases $|q| > 1$, $|q| < 1$ to the other; the theory of the equation for $|q| = 1$ differs essentially from that for $|q| \neq 1$ and unless otherwise specified it is to be understood throughout this paper that $|q|$ is $\neq 1$. The cases in which the coefficients $a_i(x)$ and $b(x)$ are of a certain character at $x = 0$ and of the same character at $x = \infty$ are reducible each to the other by the substitution $x' = 1/x$; we shall therefore in general restrict ourselves to the

* Pincherle, *Ricerche sopra una classe importante di funzioni monodrome*, Giornale di Matematiche, vol. 18 (1880), pp. 92–136.

† Stridsberg, *Contributions à l'étude des fonctions algébriques-transcendentes qui satisfont à certaines équations fonctionnelles algébriques*, Arkiv för Matematik, Astronomi och Fysik, vol. 6, no. 15, 1910, 31 pp.; vol. 6, no. 18, 1910, 25 pp. See also Carmichael, *The general theory of linear q -difference equations*, American Journal of Mathematics, vol. 34 (1912), pp. 147–168.

‡ The name *geometric difference equation* has been suggested; see Ryde, *A contribution to the theory of linear homogeneous geometric difference equations (q -difference equations)*, (Dissertation, Lund), 1921, 45 pp.

former. The variable x is in general understood to be complex, and the equation

$$(3) \quad a_0(0)\rho^n + a_1(0)\rho^{n-1} + \cdots + a_{n-1}(0)\rho + a_n(0) = 0,$$

is commonly called the *characteristic equation* of (2h) for $x=0$.

It should be observed that if we define

$$\Delta^0 f(x) = f(x), \quad \Delta f(x) = f(qx) - f(x), \quad \Delta^i f(x) = \Delta(\Delta^{i-1} f(x)),$$

then, precisely as in the case of the usual finite difference operator, $\Delta^n f(x)$ is expressible* linearly (with constant coefficients) in terms of $f(q^i x)$, ($i=0, 1, \dots, n$), and likewise $f(q^n x)$ linearly in terms of $\Delta^i f(x)$, ($i=0, 1, \dots, n$). Hence a linear equation† in $\Delta^i f(x)$, ($i=0, 1, \dots, n$), may always be written in the form (2), and vice versa.

One may properly say that the q -difference equation (2) in general presents a simpler problem than does the difference equation (1). The notion of asymptotic form, so necessary for the analytic theory of (1), has as yet played no role in that of (2). It may therefore appear a little surprising that the development of the theory of (2) in any general sense preceded by only a short space the beginning of the analytic theory of (1), which marked the reawakening of an interest, long quiescent, in the subject of difference equations.

2. Periodic Functions.

The equation

$$(4) \quad \Delta f(x) = 0,$$

remarked by Babbage‡ in 1815, is among the first q -difference equations to appear in the literature. Its solutions are naturally of primary importance in the theory of equation (2), since func-

* This fact appears to have been observed first by Thomae, *Les séries Heineennes supérieures, ou . . .*, Annali di Matematica, (2), vol. 4 (1870–71), pp. 105–138.

† Equations of this kind were styled “difference equations” at least as early as 1847; see Heine, *Untersuchungen über die Reihe*

$$1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)} x + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})} x^2 + \dots,$$

Journal für Mathematik, vol. 34 (1847), pp. 285–328.

‡ Babbage, *An essay towards the calculus of functions*, Royal Society of London, Philosophical Transactions, 1815, Part 2, pp. 389–423.

tions of this character but otherwise arbitrary play for (2) the role of arbitrary constants in the theory of linear differential equations. These "multiplicatively periodic" functions also are of importance for the theory of the generalized hypergeometric series studied by Heine* and Thomae† and of the "higher Heine series,"

$$(5) \quad \begin{aligned} 1 + & \sum_{n=1}^{\infty} \frac{1 - q^a}{1 - q} \frac{1 - q^{a+1}}{1 - q^2} \cdots \frac{1 - q^{a+n-1}}{1 - q^n} \cdot \frac{1 - q^{a'}}{1 - q^{b'}} \frac{1 - q^{a'+1}}{1 - q^{b'+1}} \\ & \cdots \frac{1 - q^{a'+n-1}}{1 - q^{b'+n-1}} \cdots \frac{1 - q^{a^{(h)}}}{1 - q^{b^{(h)}}} \frac{1 - q^{a^{(h)}+1}}{1 - q^{b^{(h)}+1}} \\ & \cdots \frac{1 - q^{a^{(h)}+n-1}}{1 - q^{b^{(h)}+n-1}} x^n, \end{aligned}$$

investigated by Thomae.‡ The properties of the solutions of (4) were made the object of a rather thorough study by Pincherle§ in 1880; these functions were also treated by Rausenberger|| in 1884. In the following pages the term "periodic function" is to be understood as meaning a solution of (4).

3. *q-Finite Integration.* The problem of *q*-finite integration, that is, of solving the equation $\Delta f(x) = b(x)$, was examined by Goursat¶ in 1903–4 in connection with his investigation of a problem in the theory of partial differential equations. Goursat assumed $b(x)$ analytic for $|x| < R$ and determined, by the method of power series with direct convergence proofs, under what conditions there exists a solution of like character. He

* See the second footnote on p. 363.

† Thomae, *Beiträge zur Theorie der durch die Heinesche Reihe:*

$1 + \frac{1 - q^a}{1 - q} \cdot \frac{1 - q^b}{1 - q^c} x + \frac{1 - q^a}{1 - q} \cdot \frac{1 - q^{a+1}}{1 - q^2} \cdot \frac{1 - q^b}{1 - q^c} \cdot \frac{1 - q^{b+1}}{1 - q^{c+1}} x^2 + \dots$

darstellbaren Functionen, Journal für Mathematik, vol. 70 (1869), pp. 258–281.

‡ See the first footnote on p. 363.

§ See the first footnote on p. 362.

|| Rausenberger, *Lehrbuch der Theorie der Periodischen Functionen . . .*, 1884, 8+476 pp.; especially pp. 221 ff.

¶ Goursat, *Sur un problème relatif à la théorie des équations aux dérivées partielles du second ordre*, Toulouse Annales, (2), vol. 5 (1903), pp. 405–436; vol. 6 (1904), pp. 117–144.

observed that for such a solution to exist $b(0)$ must vanish, and proved that in the case of $b(0) = 0$ and $|q| \neq 1$ there exists a solution analytic for $|x| < R$. For $|q| = 1$ two cases must be distinguished, according as q is or is not a root of unity. In the first let m be the least integer for which $q^m = 1$; then a solution exists if and only if the powers of x^m in the Maclaurin expansion of $b(x)$ all have coefficients zero. Moreover, when a solution exists it is determined only up to an additive arbitrary function of x^m analytic for $|x| < R$. In the second case a solution may or may not exist, but this depends upon the nature of $\arg q$ rather than upon $b(x)$. Goursat also gave a supplementary consideration to the problem in the real domain. It may be remarked that definite and indefinite q -finite integrals of certain elementary functions were found by F. H. Jackson* from 1910 to 1917.

Further light will be shed on this problem in a later section.

4. Homogeneous Equations. Case of $a_0(0) \neq 0, a_n(0) \neq 0$. The equation (2h) of order $n > 1$ was at first studied only under very restrictive hypotheses. In 1847 Heine† showed that a particular equation of the second order is satisfied by the "Heine series." It was some years later, in 1870–71, that Thomae‡ went further in proving that the equation (2h) of order n whose coefficients are linear in x and whose characteristic equation has no infinite or zero roots can be solved in terms of the series (5) of order $h = n$. In 1909–11 Jackson§ found solutions for several particular equations (2), mainly of the second order and homogeneous; most of his solutions also were expressed in power series of the generalized hypergeometric type.

* Jackson, *Borel's integral and q -series*, Proceedings of the Royal Society of Edinburgh, vol. 30 (1910), pp. 378–385; *A q -generalization of Abel's series*, Palermo Rendiconti, vol. 29 (1910), pp. 340–346; *q -difference equations*, American Journal of Mathematics, vol. 32 (1910), pp. 305–314; *On q -definite integrals*, Quarterly Journal of Mathematics, vol. 41 (1910), pp. 193–203; *The q -integral analogous to Borel's integral*, Messenger of Mathematics, vol. 47 (1917), pp. 57–64.

† See the second footnote on p. 363.

‡ See the first footnote on p. 363.

§ Jackson, *Generalization of the differential operative symbol with an extended form of Boole's equation*, Messenger of Mathematics, vol. 38 (1909), pp. 57–61; *q -difference equations*, American Journal of Mathematics, vol. 32 (1910), pp. 305–314; *The products of q -hypergeometric functions*, Messenger of Mathematics, vol. 40 (1911), pp. 92–100.

Let us now assume the coefficient functions $a_i(x)$ in (2h) to be analytic at $x=0$ and hence expressible in the form

$$a_i(x) = a_{i0} + a_{i1}x + a_{i2}x^2 + \cdots \text{ for } |x| < R, \quad (i = 0, 1, \dots, n);$$

the characteristic equation (3) may then be written

$$(6) \quad a_{00}\rho^n + a_{10}\rho^{n-1} + \cdots + a_{n-1,0}\rho + a_{n0} = 0.$$

We distinguish two essentially different cases according as the roots of (6) are or are not all finite and different from zero; for the present we confine ourselves to the former, assuming $a_{00} \neq 0$, $a_{n0} \neq 0$.

Let $\rho_j = q^{r_j}$ ($j = 1, 2, \dots, p$) be any set of p roots ($p \geq 1$) of equation (6) satisfying the following conditions: (a)

$$r_j = r_1 - m_j, \quad (j = 2, 3, \dots, p),$$

where each m_j is a positive integer or zero and the subscripts of the r 's are so chosen as to make

$$0 \leq m_2 \leq m_3 \leq \cdots \leq m_p;$$

(b) no root of (6) outside this set is equal to q^{r_1+m} for m a positive or negative integer or zero. Then there correspond to the roots of this set p formal solutions of (2h):

$$(7) \quad \begin{aligned} & x^{r_1}P(x), \\ & x^{r_2}[P(x) + tP(x)], \\ & x^{r_3}[P(x) + tP(x) + t^2P(x)], \\ & \cdot \quad \cdot \\ & x^{r_p}[P(x) + tP(x) + t^2P(x) + \cdots + t^{p-1}P(x)], \end{aligned}$$

where we have set

$$(8) \quad t = \frac{\log x}{\log q},$$

and where $P(x)$ is used in a generic sense to denote a power series in x . The coefficients in the several power series $P(x)$ can be found immediately by the method of undetermined coefficients, upon substituting the series (7) in (2h). Since all the roots of (6) can clearly be grouped in sets satisfying the conditions (a) and (b), a set of n formal solutions of (2h) is thus obtained.

Each of the power series $P(x)$ converges for x in the vicinity of the origin, and therefore the existence of n solutions of (2h) analytic in this neighborhood follows. The existence and convergence of these formal solutions is an immediate consequence of a theorem of Grévy,* who in 1894, basing his work on certain earlier studies by Königs, investigated a functional equation which includes (2h) as a particular case. It also follows from Grévy's work that no identical linear relation, with analytic coefficients satisfying the equation (4) and not all identically zero, exists between these n solutions; hence they are said to constitute a fundamental set for the equation (2h).

When no root of the characteristic equation (6) is equal to an integral power of q (including q^0) times another, so that none of the formal series solutions contain logarithmic terms, the convergence of the series and the consequent existence of analytic solutions follows also from results obtained in 1912 by Carmichael.† He employed a method of successive approximation analogous to that which he had already used in studying difference equations,‡ and gave the first treatment of the equation (2h) of order n as such with coefficients other than linear in x . He remarked that the method of Birkhoff§ for difference equations can also be applied in the case considered; not only is this true, but the same results can be obtained in this way with considerably less labor. Carmichael also investigated the case of $|q| = 1$, showing that in general analytic solutions do not exist and examining their nature when they do exist.

In 1915, T. E. Mason|| proved that if $|q| > 1$ and the coefficients $a_i(x)$ are entire functions, with $a_0(x) \equiv 1$, and if the roots of (6) are not all zero, there exist one or more solutions of the form $f(x) = x^{r_i} E_i(x)$, where $E_i(x)$ is entire. This result was obtained by reckoning out the coefficients in the formal series

* Grévy, *Étude sur les équations fonctionnelles*, (Dissertation, Paris), 1894; reprinted in large part in *Annales de l'École Normale Supérieure*, (3), vol. 11 (1894), pp. 249–323.

† See the second footnote on p. 362.

‡ Carmichael, *Linear difference equations and their analytic solutions*, Transactions of this Society, vol. 12 (1911), pp. 99–134.

§ Birkhoff, *General theory of linear difference equations*, Transactions of this Society, vol. 12 (1911), pp. 243–284.

|| Mason, *On properties of linear q -difference equations with entire function coefficients*, American Journal of Mathematics, vol. 37 (1915), pp. 439–444.

solutions and proving their convergence directly, and it was shown that the order of the entire function $E_i(x)$ is not greater than the maximum order of the coefficients.

Direct proof of the convergence of the formal series (7) under all conditions was given by Adams* in 1929. This method appears to have some advantage in brevity and simplicity over the others which have been used for establishing the existence of analytic solutions. It may be remarked that Mason's results can now immediately be extended to show that if the coefficients are entire functions, with $a_0(x) \equiv 1$ and $|q| > 1$, each of the power series in (7) converges for all finite values of x and represents an entire function whose order is not greater than the maximum order of the coefficients.

For brevity let us assume for the moment that $|q|$ is > 1 . Then if the coefficients $a_i(x)$ have only isolated singularities in the finite plane and $a_0(x)$ vanishes only at isolated points, it is clear that the solutions can be analytically continued away from $x=0$ indefinitely by repeated use of the equation (2h) itself. Thus it appears that the nearest singularity of any solution to the origin is no nearer than the nearest singularity of the functions

$$(9) \quad \frac{a_i(x)}{a_0(x)}, \quad (i = 1, 2, \dots, n),$$

and hence the radius of convergence of each power series in the set of formal solutions is at least as great as the distance to the nearest singularity of the functions (9). Moreover the singularities are isolated (occurring at no points other than $q^m x_i$ where m is a positive integer and x_i is any singularity of one or more of the functions (9)) and if the functions (9) have no singularities other than poles, as is the case when the $a_i(x)$ are rational functions, then the solutions will be analytic except for poles in the finite plane away from $x=0$. Similar statements hold when $|q|$ is < 1 , except that $a_0(x)$ is replaced by $a_n(x)$.

If the functions $a_i(x)$ are analytic both at $x=0$ and $x=\infty$ and if the roots of the characteristic equations for both points satisfy the restriction imposed by Carmichael, the equation (2h) has two sets of fundamental solutions, one of the form $x^{r_i} A_i(x)$ with

* Adams, *On the linear ordinary q-difference equation*, Annals of Mathematics, (2), vol. 30 (1929), pp. 195–205.

$A_i(x)$ analytic at $x=0$ and the other of like form with $A_i(x)$ analytic at $x=\infty$. Each set of solutions may then be expressed linearly in terms of the other with coefficients $P_{ij}(x)$ satisfying the equation (4). Employing a system of n equations of the first order rather than a single equation (2h) of order n , Carmichael* examined these periodic functions and showed that those by means of which the solutions associated with $x=0[\infty]$ are expressed in terms of the solutions associated with $x=\infty[0]$ for $|q|>1[<1]$ are analytic away from the points 0 and ∞ when the coefficient functions are polynomials in x and analytic save for poles when the coefficients are rational functions of x .

Necessary and sufficient conditions that the equation (2h) of first order have a rational solution were found by Mason† in 1914. He also showed that the solutions of the equation

$$f(qx) - (1+x)f(x) = 0$$

are transcendently transcendental.

In 1918 Carmichael‡ discussed what he termed "repeated solutions" of a linear homogeneous equation involving a general operator D which includes the differential operator, the ordinary difference operator, and the q -difference operator as particular cases. By definition $f(x)$ is said to be an r -fold solution when the functions $x^i f(x)$ ($i=0, 1, \dots, r-1$) all satisfy the equation but $x^r f(x)$ does not.§ Carmichael's chief result for q -difference equations is that a necessary and sufficient condition that a function $f(x)$ be a "repeated solution" of a linear homogeneous q -difference equation of order n (expressed in terms of $\Delta^i f(x)$, ($i=0, 1, \dots, n$)) is that it satisfy both the equation itself and the equation obtained from it by formal differentiation with respect to Δ and the replacement of x by x/q in the coefficient functions.

* See the second footnote on p. 362.

† Mason, *Character of the solutions of certain functional equations*, American Journal of Mathematics, vol. 36 (1914), pp. 419-440.

‡ Carmichael, *Repeated solutions of a certain class of linear functional equations*, Tôhoku Mathematical Journal, vol. 13 (1918), pp. 304-313.

§ That this terminology has a reasonable basis when D represents the differential or difference operator is manifest, but it seems inappropriate when D stands for the q -difference operator, since an equation (2h) with constant coefficients having $f(x)$ as a particular solution corresponding to a multiple root of the characteristic equation has $(\log x)f(x)$ as a second solution rather than $xf(x)$.

He also obtained such a condition in terms of the coefficients alone.

An existence proof quite different from those cited above was given in 1921 by Ryde,* who employed the “geometric factorial series”†

$$(10) \quad a_0 + \sum_{s=1}^{\infty} \frac{a_s}{(1-x)(1-x/q)\cdots(1-x/q^{s-1})}.$$

He showed first that a necessary and sufficient condition that a given function be developable in a series of this type is that this function be analytic at $x = \infty$. The coefficients in (2h) were then assumed to be expressed in terms of series of type (10) and it was proved that the equation is formally satisfied by series of the same kind, multiplied by suitable Heinean functions. The existence of analytic solutions was shown by proving the convergence of the formal series.

In 1922 Carmichael‡ indicated how the study of systems of algebraic equations may be used as a guide in determining the properties of solutions of q -difference equations, ordinary and partial, and of integro- q -difference equations, to which one may pass from the algebraic systems by suitably chosen limiting processes. The independent variable x is assumed to be real and the problems of oscillation, comparison, and expansion are the ones specially considered. A somewhat detailed treatment of these problems, namely, oscillation properties of a fundamental system of solutions of (2h) of order 2 and of order $n > 2$, comparison properties for the solutions of two particular equations (2h) of order 2, and expansion properties for adjoint systems of order n containing a parameter linearly—was made in a second paper by the same author.§

* See the third footnote on p. 362.

† This follows the lines of Nörlund, *Sur l'intégration des équations linéaires aux différences finies par des séries de facultés*, Palermo Rendiconti, vol. 35 (1913), pp. 177–216.

‡ Carmichael, *Algebraic guides to transcendental problems*, this Bulletin, vol. 28 (1922), pp. 179–210.

§ Carmichael, *Boundary value and expansion problems: oscillation, comparison, and expansion theorems*, American Journal of Mathematics, vol. 44 (1922), pp. 129–152. See also the third footnote on p. 361.

5. Generalized Riemann Problem. In 1913 Birkhoff* formulated and solved what may be called the generalized Riemann problem for linear q -difference equations. He dealt with a system of n equations of the first order rather than a single equation of the n th order, and assumed that all the formal series solutions are free from logarithms. As a preliminary step he completely determined the periodic functions by means of which the solutions associated with $x=0$ are expressed in terms of those associated with $x=\infty$ when the coefficients are polynomials. Denoting the respective matrices of solutions by

$$F_0(x) = (x^\rho i f_{ij}(x)), \quad F_\infty(x) = q^{(\mu/2)(t^2-t)}(x^{-\sigma_i} g_{ij}(x)),$$

where μ denotes the maximum degree of the polynomial coefficients of the system and t is given by (8), and setting

$$F_0(x) = F_\infty(x)P(x), \quad P(x) = (p_{ij}(x)),$$

he showed that $p_{ij}(x)$ has the form

$$c_{ij} \left(\exp \left\{ -\frac{\eta\mu}{2} t^2 + \left[\eta(\sigma_i + \rho_j) - \frac{\eta'\mu}{2} \right] t \right\} \right) \sigma(t - \alpha_1^{(i,j)}) \\ \cdots \sigma(t - \alpha_\mu^{(i,j)}),$$

where $\sigma(t)$ is the Weierstrass sigma function associated with the periods 1 and $2\pi(-1)^{1/2}/\log q$ and

$$(11) \quad \sum_{\lambda=1}^{\mu} \alpha_\lambda^{(i,j)} = \sigma_i + \rho_j - \mu\pi(-1)^{1/2}/\log q.$$

The $2n+n^2(\mu+1)$ numbers

$$(12) \quad \rho_j, \sigma_j, c_{ij}, \alpha_1^{(i,j)}, \dots, \alpha_\mu^{(i,j)} \quad (i, j = 1, 2, \dots, n),$$

of which only $n^2\mu+1$ are independent, were designated "characteristic constants," and it was proved that for any assigned set of constants (12) subject to the condition (11) there exists a system having this set of characteristic constants or one differing from it only in the replacement of σ_i by σ_i+l_i , where the l_i are integers.

* Birkhoff, *The generalized Riemann problem* . . . , Proceedings of the American Academy of Arts and Sciences, vol. 49 (1913), pp. 521–568.

It would be of considerable interest to extend these results to the case in which some of the formal series solutions contain logarithms.

6. *Non-homogeneous Equations.* In 1915 it was shown by Mason* that the equation $(2n)$, with entire function coefficients, $|q| > 1$, and $a_0(x) \equiv 1$, has an entire function solution whenever it has a formal power series solution, and that the order of the solution does not exceed the maximum order of the coefficients.

The first study of the non-homogeneous q -difference equation under general hypotheses appears to have been made by Adams† in 1925. He employed a system of n equations of the first order in preference to the equation $(2n)$ and showed that in a large class of cases, when the coefficients have the character of rational functions at $x=0$, the system is satisfied formally by a series which converges and so represents an analytic solution. The method employed makes use of the notion of "variation of the constants" and of an evaluation for the operator $\sum = \Delta^{-1}$ in terms of series. When the coefficients are of such character that the system admits two solutions, one associated with $x=0$ and the other with $x=\infty$, the relation between the solutions was studied.

The equation $(2n)$ itself was investigated by the same author‡ in 1929 on the assumption that the coefficients $a_i(x)$ and $b(x)$ are all analytic at $x=0$. Here again the existence of analytic solutions was shown by exhibiting series that formally satisfy the equation and proving their convergence directly. These results may be generalized immediately to the case in which $b(x)$ in $(2n)$ has the form

$$(13) \quad b(x) = \sum_{i=0}^m t^i b_i(x^{1/s}),$$

where t is given by (8), s is a positive integer ≥ 1 , and each function $b_i(x^{1/s})$ is an analytic function of $x^{1/s}$ at $x=0$. For our later purposes it will be convenient to have the following statement of this generalization.

* See the fifth footnote on p. 367.

† Adams, *Note on the existence of analytic solutions of non-homogeneous linear q-difference equations, ordinary and partial*, Annals of Mathematics, (2), vol. 27 (1925), pp. 73–83; vol. 30 (1929), p. 626.

‡ See the footnote on p. 368.

If among the quantities q^i , ($i = 0, 1, 2, \dots$), there are exactly p roots ($p \geq 0$) of the equation (6), each root being counted according to its multiplicity, the equation (2n) with $b(x)$ given by (13) has a formal solution

$$(14) \quad s(x) = P(x^{1/s}) + tP(x^{1/s}) + \dots + t^{m+p}P(x^{1/s}),$$

in which the coefficients in the power series $P(x^{1/s})$ may be calculated by substitution in the equation (2n). If $|q|$ is > 1 [< 1] and the equation (6) has no infinite [zero] roots, the series in (14) converge in the vicinity of $x=0$.

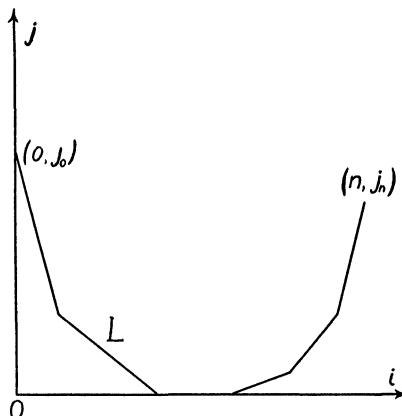
It is of importance to observe that this result gives a q -finite integral of every function of the type (13). One should note also that Mason's results* may immediately be generalized as follows: if $|q|$ is > 1 and the coefficients $a_i(x)$ of (2n) are entire functions with $a_0(x) \equiv 1$, and if $b(x)$ is of the form (13) with each $b_j(x^{1/s})$ an entire function of $x^{1/s}$, then each $P(x^{1/s})$ in (14) converges for every finite value of $x^{1/s}$ and represents an entire function of $x^{1/s}$ whose order does not exceed the maximum order of the $a_i(x)$ and $b_j(x^{1/s})$.

7. Homogeneous Equations. Case of $a_0(0)=0$ or $a_n(0)=0$ or Both. In this case also we assume each coefficient $a_i(x)$ to be analytic at $x=0$ and so expressible in the form $\sum_{j=0}^{\infty} a_{ij}x^j$ for $|x| < R$. Let a_{i,j_i} denote the first non-zero coefficient in $a_i(x)$ and plot the points (i, j_i) in the ij -plane. Then construct a broken line L , convex downward, such that both ends of each segment are points of the set (i, j_i) while all points of the set lie on or above this line. No restriction is imposed by assuming at least one point of the set to be on the i -axis, for otherwise an integral power of x could be suppressed from the entire equation.

The equation (6) is now by no means completely characteristic of the difference equation (2h). In fact one might say that (3) is replaced by several characteristic equations, one associated with each segment of L . The degree of the characteristic equation associated with any segment of L is one less than the number of points that lie on or vertically above that segment, including its end points. The coefficients of this characteristic equation are the a_{i,j_i} corresponding to points (i, j_i) actually on that segment; the coefficient corresponding to a point (i, j_i) ver-

* See the fifth footnote on p. 367.

tically above the segment is zero. Evidently the sum of the degrees of these several characteristic equations is n . If one of the segments is horizontal, the characteristic equation associated with that segment may be obtained from (6) by suppressing the



infinite and zero roots. To the roots of this equation there correspond formal solutions of the type (7), in number equal to the degree of the equation. In order to obtain solutions associated with a segment of slope $\mu \neq 0$ we make the transformation

$$(15) \quad f(x) = q^{(\mu/2)(t^2-t)} g(x).$$

This changes the equation (2h) into a new equation which, after the suppression of factors common to all its terms, is of exactly the same type except in the respect that if μ is not an integer but a fraction, r/s in lowest terms and s positive, certain of the power series coefficients are multiplied by positive integral powers of $x^{1/s}$. The effect of (15) upon the points (i, j_i) is to relocate them in such a way that each segment of the new broken line L' , similarly constructed for the transformed equation, has a slope μ less than that of the corresponding segment of L . Thus, in particular, the segment of L whose slope is μ becomes a horizontal segment of L' .

When μ is an integer there corresponds to the segment of L having this slope a set of formal solutions exactly like (7) except for the additional factor $q^{(\mu/2)(t^2-t)}$ in each one. To a segment whose slope is a fraction, r/s , there corresponds a set of formal solutions like (7) in type except that each contains the addi-

tional factor $q^{(\mu/2)(t^2-t)}$ and each $P(x)$ is replaced by a series in powers of the s th root of x , $P(x^{1/s})$.

A full set of n formal solutions is thus obtained for the equation (2h). These were exhibited by Adams* in 1929, together with direct proof of the convergence, in the vicinity of $x=0$, of the series associated with the segment of L farthest to the left [right] when $|q|$ is >1 [<1]. The type of convergence proof used is clearly not applicable to the series associated with other segments of L ; indeed examples in which the series associated with other segments converge only for $x=0$ are easily constructed. Thus the question of whether there always exist analytic solutions associated with the other segments was left open. Some light, however, is shed on the situation by the following discussion, in which we assume for brevity of statement that $|q|$ is >1 .

It is clearly no real restriction to suppose that the segment farthest to the left in the broken line L belonging to the equation (2h) is horizontal. The equation then has at least one formal solution free from logarithms,

$$s_1(x) = x^{r_1}(s_1 + s'_1 x^{1/s} + s''_1 x^{2/s} + \dots), \quad q^{r_1} = \rho_1 \neq 0,$$

where s is an integer ≥ 1 , and this series converges for $x^{1/s}$ in the neighborhood of $x=0$. We propose to employ this solution to reduce the order of (2h); to this end let us set

$$f(x) = s_1(x) \sum g(x), \quad \sum = \Delta^{-1}.$$

Substituting in (2h) we find

$$(16) \quad \begin{aligned} a_0(x)s_1(q^n x)g(q^{n-1}x) + & \left[\sum_{i=0}^1 a_i(x)s_1(q^{n-i}x) \right]g(q^{n-2}x) + \dots \\ & + \left[\sum_{i=0}^{n-1} a_i(x)s_1(q^{n-i}x) \right]g(x) = 0, \end{aligned}$$

for which the equation analogous to (6) is

$$(17) \quad a_{00}\rho_1^n \sigma^{n-1} + \left[\sum_{i=0}^1 a_{i0}\rho_1^{n-i} \right] \sigma^{n-2} + \dots + \left[\sum_{i=0}^{n-1} a_{i0}\rho_1^{n-i} \right] = 0.$$

Let us now assume for the moment that the equation (6) has

* See the first footnote on p. 368.

exactly m roots ($0 \leq m \leq n-2$) equal to zero, so that it may be written

$$a_{00}\rho^n + a_{10}\rho^{n-1} + \cdots + a_{n-m,0}\rho^m = 0, \quad a_{n-m} \neq 0.$$

The coefficient of σ^m in (17) is then precisely

$$-a_{n-m,0}\rho_1^m \neq 0,$$

while the subsequent coefficients are all zero; thus the equation (17) has exactly m zero roots and consequently one less non-zero root than (6). Moreover if σ_1 is any non-zero root of (17), $\rho_1\sigma_1$ is a root of (6). For we may write (17) for $\sigma = \sigma_1$ and multiply by σ_1 to obtain

$$(18) \quad a_{00}\rho_1\sigma_1^n + a_{10}\rho_1^{n-1}\sigma_1^{n-1} + a_{20}\rho_1^{n-2}\sigma_1^{n-2} + \cdots + a_{n-1,0}\rho_1\sigma_1 + a_{00}\rho_1\sigma_1^n + \left[\sum_{i=0}^1 a_{i0}\rho_1^{n-i} \right] \sigma_1^{n-2} + \cdots + \left[\sum_{i=0}^{n-2} a_{i0}\rho_1^{n-i} \right] \sigma_1 = 0.$$

Since σ_1 satisfies (17) the sum of terms in the second line equals

$$- \sum_{i=0}^{n-1} a_{i0}\rho_1^{n-i},$$

which, ρ_1 being a root of (6), is equal to a_{n0} . Thus (18) expresses the fact that $\rho_1\sigma_1$ is a root of (6).

The equation (16) has at least one formal solution

$$s_2(x) = x^{r_2}(s_2 + s_2' x^{1/s} + s_2'' x^{2/s} + \cdots), \quad q^{r_2} = \sigma_1 \neq 0,$$

and the series converges for $x^{1/s}$ in the neighborhood of $x=0$. From §6 it follows that a q -finite integral of $s_2(x)$ is

$$\sum s_2(x) = x^{r_2}[P(x^{1/s}) + tP(x^{1/s}) + \cdots + t^m P(x^{1/s})],$$

each $P(x^{1/s})$ being convergent in the vicinity of $x=0$ and m an integer ≥ 0 . Hence

$$s_1(x) \sum s_2(x)$$

is a solution of (2h), and since $q^{r_1+r_2} = \rho_1\sigma_1$ is a root of (6) this solution must be essentially the same as a solution already known to exist. This process of reducing the order can evidently be carried out repeatedly and a complete set of solutions associated with the segment of L farthest to the left can thus be ob-

tained. If L consists of more than one segment, the reduced equation, after a finite number of reductions, will have a characteristic equation of type (6) all of whose roots are zero. For a suitable value of μ the reduced equation will have at least one solution expressible in the form

$$(19) \quad q^{(\mu/2)(t^2-t)} x^{r_3} (s_3 + s'_3 x^{1/s} + s''_3 x^{2/s} + \dots)$$

for $x^{1/s}$ in the vicinity of $x=0$. This method of obtaining solutions of (2h) therefore must stop at this point unless a q -finite integral of the function (19) can be determined. Indeed we can proceed further if and only if we can "integrate" functions of the following character,

$$(20) \quad q^{(\mu/2)(t^2-t)} x^r t^m a(x^{1/s}),$$

$a(x^{1/s})$ being an analytic function of $x^{1/s}$ in the vicinity of $x=0$ and m an integer ≥ 0 .

If the coefficient functions $a_i(x)$ are meromorphic over the entire finite plane, the function $a(x^{1/s})$ will be a meromorphic function of $x^{1/s}$ in the same region. In this case we can determine a q -finite integral of (20), or in other words a solution of the equation

$$(21) \quad h(qx) - h(x) = q^{(\mu/2)(t^2-t)} x^r t^m a(x^{1/s}),$$

as follows. Let the variables be changed by setting

$$x = q^z, \quad h(q^z) = \phi(z);$$

then (21) becomes the difference equation

$$(22) \quad \begin{aligned} \phi(z+1) - \phi(z) &= q^{(\mu/2)(z^2-z)} q^{rz} z^m a(q^{z/s}) \\ &= e^{(\mu/2)(z^2-z) \log q + rz \log a z^m a(e^{(z/s) \log q})}. \end{aligned}$$

The first two factors on the right are entire functions; the third has no singularities in the finite z -plane other than poles. Hence the right-hand member of (22) is meromorphic over the entire finite z -plane and there exists* a meromorphic sum of this function, that is, a meromorphic solution of the equation (22).

When a full set of n solutions of the equation (2h) can be obtained by the method indicated, it would be of interest to show

* See Hurwitz, *Sur l'intégrale finie d'une fonction entière*, Acta Mathematica, vol. 20 (1897), pp. 285-312.

that these solutions constitute a fundamental set and to exhibit the relation, if one exists, between the solutions associated with segments of L other than that farthest to the left and the formal series solutions associated with those segments. It would also be of value to discuss the generalized Riemann problem in this case and to investigate the properties of the solutions when the coefficients $a_i(x)$ are entire functions.

8. *Expansion Problems.* A detailed study of expansion problems connected with solutions of the equation (2h) was made by Carman* in 1926. He assumed all the coefficients $a_i(x)$ analytic at $x=0$ and introduced a parameter λ_m linearly into $a_n(x)$. It was shown first that for each integer $m \geq 0$, λ_m can be determined so that the equation will be satisfied by a function

$$u_m(x) = x^{m+\nu} \left(1 + \sum_{j=1}^{\infty} c_{mj} x^j \right),$$

where ν is any constant and the series converges in the vicinity of $x=0$. Then followed a demonstration that x^p (p an integer ≥ 0), and likewise any given function analytic at $x=0$, can be expanded in a series

$$\sum_{m=0}^{\infty} c_m U_m(x), \quad U_m(x) = x^{-\nu} u_m(x),$$

uniformly convergent in a circle about $x=0$. Carman established a similar expansion theorem for a function analytic at $x=\infty$, developed an analog of the Laurent series, and proved a number of orthogonality conditions for the sets of functions $U_m(x)$ and $u_m(x)$, the integrals being taken over a closed path in the vicinity of $x=0$ encircling that point once. Generalizations of these results for functions of several variables were also pointed out.

9. *Partial q -Difference Equations.* Linear partial q -difference equations have been studied by Adams in four papers from 1924 to 1929. In the first of these† the equation

* Carman, *Expansion problems in connection with homogeneous linear q -difference equations*, Transactions of this Society, vol. 28 (1926), pp. 523–535.

† Adams, *The general theory of a class of linear partial q -difference equations*, Transactions of this Society, vol. 26 (1924), pp. 283–312; the contents of this paper, as well as that referred to in the fourth footnote on p. 379, are contained

$$(23) \quad \sum_{i=0}^n a_i(x, y) f(q^{n-i}x, r^{n-i}y) = 0$$

with coefficients $a_i(x, y)$ analytic at $(0, 0)$ was considered. Solutions were found by obtaining them for the related system of n equations of the first order, following the method of Birkhoff* already cited. When the system possesses n sets of formal solutions in powers of x and y (as will be the case when the characteristic equation for $(0, 0)$ has no zero or infinite roots and no root equal to another times $q^a r^b$ for integral a and b) and when both $|q|$ and $|r|$ are greater or less than 1, these series converge in the vicinity of $(0, 0)$ and represent analytic solutions there. The equation of first order whose characteristic equation has the root zero was also examined in this paper. It was shown that for $|q| = |r| = 1$ the equation in general admits no analytic solution that is not identically zero. When the coefficients $a_i(x, y)$ are rational functions, the fundamental periodic functions by means of which one set of solutions is expressed in terms of another (for example, the solutions associated with $(0, 0)$ and with (∞, ∞) when both $|q|$ and $|r|$ are greater or less than 1) are analytic away from $x=0, x=\infty, y=0, y=\infty$; moreover these functions are triply periodic and of the class studied by Cousin† in 1910.

In the second paper‡ Adams proved for the non-homogeneous equation of type (23) results analogous to those already cited for the equation $(2n)$. The third paper§ was devoted to the mixed equation

$$f(x + 1, ry) = a(x, y)f(x, y),$$

where $a(x, y)$ is a polynomial with constant term not zero. The existence of principal solutions was established by the method of Birkhoff,|| and the relation between two such, when they exist, was examined.

in essentially the same form in *The general theory of the linear partial q-difference equation and of the linear partial difference equation of the intermediate type*, (Dissertation, Harvard), 1922.

* See the fourth footnote on p. 367.

† Cousin, *Sur les fonctions triplement périodiques de deux variables*, Acta Mathematica, vol. 33 (1910), pp. 105–232.

‡ See the second footnote on p. 372.

§ Adams, *Existence theorems for a linear partial difference equation of the intermediate type*, Transactions of this Society, vol. 28 (1926), pp. 119–128.

|| See the fourth footnote on p. 367.

Adams' fourth paper* treated the equation

$$(24) \quad \sum_{i,j=0}^n a_{ij}(x, y)f(q^{n-i}x, r^{n-j}y) = 0,$$

with coefficients analytic at $(0, 0)$. If we let

$$a_{ij}(0, 0) = a_{ij00},$$

the characteristic equation of (24) at $(0, 0)$ is

$$\sum_{i,j=0}^n a_{ij00} \rho^i \sigma^j = 0.$$

A pair of values (ρ_0, σ_0) satisfying this equation is called a characteristic number-pair. It was demonstrated that under general conditions there corresponds to such a number-pair a formal solution which in the general case is of the form

$$x^\mu y^\nu [(P.S.) + t(P.S.) + t^2(P.S.) + \cdots + t^m(P.S.)],$$

where $\mu = \log \rho_0 / \log q$, $\nu = \log \sigma_0 / \log r$, and where $(P.S.)$ stands in a generic sense for a power series in x and y , and t represents one or the other of the functions

$$\frac{\log x}{\log q}, \quad \frac{\log y}{\log r}.$$

The existence of analytic solutions was established by direct proof of the convergence of these formal series. The non-homogeneous equation of type (24) was also considered.

In connection with this last paper several interesting questions arise. Under certain conditions there appear to be a smaller number of formal solutions associated with a characteristic number-pair than one would expect; are there others which have not yet been exhibited? Sometimes also there appear to be a larger number than would normally be anticipated; are these in a proper sense linearly dependent? What is the most general solution of an equation of type (24)?

10. Differential q -Difference Equations. The only detailed consideration yet given to the differential- q -difference equation

* Adams, *On the linear partial q -difference equation of general type*, Transactions of this Society, vol. 31 (1929), pp. 360-371.

is by Flamant* who in 1924 made an elaborate study of the equation

$$(25) \quad f'(x) = a(x)f(qx) + b(x), \quad (|q| < 1),$$

where $a(x)$ and $b(x)$ are analytic in a closed and simply connected region D such that if x is in D , qx is likewise. It is clear that the combination of conditions placed upon $f(x)$ by (25) and by a prescribed value at $x=0$ may be expressed by an integral equation of Volterra type. By a method of successive approximation the author established the existence of a unique solution analytic in D . He indicated how the results may be generalized to a type of equation which includes as a particular case the equation obtained from (25) by replacing $a(x)f(qx)$ by a linear combination of the functions $f(q^i x)$ ($i=1, 2, \dots, n$) with coefficients analytic in D . The greater portion of the paper was devoted to examining the effect upon the solutions of permitting $b(x)$ to have singularities in D . For example, a pole of order $\mu > 1$ [$= 1$] of $b(x)$ at $x=x_0 \neq 0$ impresses upon the solution of (25) poles of order $\mu - 1$ [logarithmic singularities] at the points x_0/q^n of the region D . When logarithmic singularities are present it is possible to develop the solution in a series involving logarithms. The case of $|q|=1$ was also given detailed treatment, as well as that in which $b(x)$ has singularities of various types at $x=0$.

11. *Integro-q-Difference Equations.* Integro-functional equations have been the object of study by several authors. Some of these equations, especially the one examined by Tamarkin,† include as a particular case an equation in which the unknown function $f(x)$ is subjected both to an integral operator of Volterra type and to the q -difference operator of first order. In 1929 Adams‡ considered the equations obtained from (2h) and (2n) by adding to the right-hand member the term

* Flamant, *Sur une équation différentielle fonctionnelle linéaire*, Palermo Rendiconti, vol. 48 (1924), pp. 135–208.

† Tamarkin, *On Volterra's integro-functional equation*, Transactions of this Society, vol. 28 (1926), pp. 426–431.

‡ Adams, *Note on integro-q-difference equations*, Transactions of this Society, vol. 31 (1929), pp. 861–867.

$$\int_0^x K(x, \xi) f(\xi) d\xi.$$

When the known functions involved are analytic at $x=0, \xi=0$, these equations are in general satisfied formally by certain power series, multiplied by suitable powers of x . If $|q|$ is > 1 the convergence of these series can be established and solutions analytic in the vicinity of $x=0$ are thereby determined. If $|q|$ is < 1 the situation is essentially different. Under further restrictions an immediate extension of Tamarkin's results establishes the existence of a unique solution analytic in the neighborhood of $x=0$ when $b(x)$ does not vanish identically, and shows that when $b(x)$ is identically zero there exists no solution bounded in the neighborhood of $x=0$. Under a different set of further restrictions, in some respects less stringent, the existence of a solution analytic at $x=0$ is demonstrated by a method of successive approximation. Special consideration is also given in this paper to the problem in the real domain. A question of interest, as yet unanswered, is: what is the most general solution of such an equation?

12. *Other Problems.* In 1930 Starcher* showed how the equation

$$xf(qx) - f(x) = 0, \quad (|q| < 1),$$

may be made the basis for the derivation of the two fundamental expressions for the θ -functions and for the deduction of their properties. In a forthcoming paper† this author also obtains a large number of identities by equating different expressions for the solution of a q -difference equation, of the first order non-homogeneous or of the second order homogeneous, with linear coefficients. Some of these identities are new; others, already given by Euler, Gauss, Cauchy, and other more recent writers, are established by a new method. Number-theoretical interpretations are given to some of the results.

* Starcher, *A solution of a simple functional equation as a basis for readily obtaining certain fundamental formulas in the theory of elliptic functions*, this Bulletin, vol. 36 (1930), pp. 577-581.

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SUPPLEMENTARY TO NÖRLUND'S BIBLIOGRAPHY OF THE CALCULUS OF FINITE DIFFERENCES AND DIFFERENCE EQUATIONS

Nörlund's *Differenzenrechnung* (1924) includes a notable bibliography, the first of any considerable importance in this field. On account of the rapid growth of the literature of difference equations and related subjects we venture to present here a list supplementing that of Nörlund. Of the following titles, numbering well over three hundred, something over ninety come within the period covered by Nörlund; the rest are of more recent publication. Except for the inclusion of a few book reviews and of a small number of abstracts of papers presented to the American Mathematical Society but apparently not yet published in full, it has been our intention so far as possible to employ the same criteria for acceptance of titles that Nörlund used. Only five of the titles have not been seen by us; these are designated by asterisks.

It may be observed that Nörlund's title, F. H. Jackson [14], should have been listed as C. S. Jackson [1]. In general, however, Nörlund's bibliography is exceedingly accurate and complete, and we hope that ours may be found a worthy and useful supplement to it.*

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* The writer wishes to express his grateful appreciation to R. C. Archibald for many valuable suggestions concerning the preparation of this list.

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