Erreurs et Moindres Carrés. By R. Deltheil. Paris, Gauthier-Villars, 1930. v+161 pp.

This treatise is the second paper of an excellent series of monographs on probability, entitled *Traité du Calcul des Probabilités et de ses Applications*, edited by Émile Borel, with the collaboration of L. Blaringhem, C. V. L. Charlier, R. Deltheil, H. Galbrun, J. Haag, R. Lagrange, F. Perrin, and P. Traynard.

On the theoretical side, Deltheil sketches in a masterly way two of the main routes rigorously leading to the so-called Gaussian probability law (Chapters 2, 4, 6, 7). The first of these is the method of moments, using staircase functions, going back to Tchebychef, Bienaymé, Stieltjes, Castelnuovo, and Liapounoff. Deltheil refers the readers to the monograph written by Borel as the first paper of the *Traité* for further details. The second rigorous method is that of Paul Lévy who uses as a "characteristic function" the Stieltjes integral of $e^{ixt}dF(x)$, where $F(x)$ is the cumulative frequency law. This characteristic function goes back to Cauchy, and still further back to Laplace.* Deltheil's treatment here is a felicitous condensation of Lévy's excellent *Calcul des Probabilités*, to which the reader is referred for details. An interesting discussion of the hypotheses required by the two methods of approach to the Gaussian Law is given (pp. 80–82), in which some later work by Fréchet † is also mentioned, whose hypotheses lead to a different law.

In any comprehensive treatment of least squares, some use or at least mention of Bayes' Theorem seems inevitable, in spite of the difficulties involved in making the treatment rigorous. So long as the a priori probability is under control, no special obstacle arises; and Deltheil's Chapter 1 gives a very satisfactory and instructive treatment for this aspect. Chapter 3 picks up Bayes' Theorem again for use with measurements where, in general, the a priori probabilities are not known and cannot be known. The hypothesis of constant a priori probability is accepted temporarily as the only reasonable assumption in case of total ignorance, but the reader is referred to Chapter 9 for release from this objectionable assumption. In Chapter 5, devoted to the principle of the arithmetic mean, the difficulty reaches its culmination. The original "demonstration" of Gauss is given, and some of the numerous objections raised thereto. Deltheil concludes (p. 63) that the critics of Gauss are justified. Poincaré's attempt is given (pp. 59–63) to salvage something from the wreck. Here, the principle of the arithmetic mean as the most probable value leads to the assumption that the a priori probability $\phi(x)$ is a constant. A rather peculiar constant, indeed, this would be; for the integral of $\phi(x)$ from $-\infty$ to $+\infty$ must be unity. Possibly, this "proof" of Gauss, abandoned by Gauss himself (p. 63), has great "historical value"; since nearly every book on least squares incorporates it—often reluctantly. But, perhaps the space devoted to such obsolete material could be better used. Instead of clinging to the untenable "$\phi(x)=$constant," it is possible to show that under very broad conditions the effect of $\phi(x)$

* E. C. Molina, *The theory of probability: some comments on Laplace's Théorie Analytique*, this Bulletin, vol. 36 (1930), pp. 369–392.

† *Sur l'hypothèse de l'additivité des erreurs partielles*, Bulletin des Sciences Mathématiques, 1928.

is evanescent as the number of measurements increases. One of the most comprehensive and far-reaching treatments along this line is that of R. von Mises,* whose results, expressible with k-dimensional vectors, are of great generality.

Deltheil does to some extent disentangle his treatment from Bayes' Theorem in subsequent chapters. Error-risk is introduced and *linear* functions of the measurements (p. 101) are considered. It is, indeed, possible to prove rigorously that under the Gaussian Law the arithmetic mean has a probability greater than that of any other linear function or *weighted mean*, in case the "precision" is constant, and to prove the analogous theorem for measurements of unequal precision. So long as we make comparisons only among linear functions, we can move on safe ground.

The subject of least squares, as a practical tool for adjustment of measurements in geodesy and kindred sciences, has a rather well-defined content. Deltheil treats in an admirable manner the topics generally required, giving numerical illustrations in considerable detail. He also sketches a few other topics such as the Gram-Charlier development in Hermite polynomials (pp. 87–90) and Poincaré's method of successive approximations (pp. 94–96). A five-place table of the probability integral concludes the volume.

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Methoden der Mathematischen Physik. Bd. I, zweite verbesserte Auflage. By R. Courant and D. Hilbert. Berlin, Springer, 1931. xiv+469 pp.

The first edition (1924) of this important and useful book has been already reviewed in a very detailed manner by E. Hille (this Bulletin, vol. 31 (1925), pp. 456–459) and besides is so well known that we may restrict ourselves here to a few remarks. Although the number of pages has not increased considerably (xiii+450 in the first edition) the number of changes is large. Practically all the chapters and sections contain modifications, in exposition as well as in the order of the material. Among the most important additions the following should be mentioned separately: proof of the completeness of the sets of Laguerre's and Hermite's polynomials (Chapter II, 9.6); transformations of problems of the calculus of variations (Chapter IV, 9) where, on the basis of a recent paper by K. Friedrichs (Göttinger Nachrichten, 1929, pp. 13–29) it is shown that in many cases a minimum problem can be transformed into an equivalent maximum problem, with the same value of the extremum in question. The occurrence of a continuous spectrum in problems of mathematical physics is illustrated, of course, by the example of Schrödinger's equation (Chapter V, 12.4). In Chapter VI, 5, a generalized Schrödinger equation is treated from the point of view of the calculus of variations. However, no satisfactory treatment of the continuous spectrum is obtained in this fashion.

The bibliographical references are a little more complete in the present edition than in the first one. In this connection the reference to an unpublished paper by R. G. D. Richardson should be welcomed (p. 404). This paper was

* *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 4 (1919), pp. 1–97.