PROBABILITY AND PHYSICAL SYSTEMS*

BY G. D. BIRKHOFF

1. *Introduction.* My aim today is to lay before you some recent developments in a mathematical field which owes its very existence to the problems of the physicist and astronomer, namely that of ordinary differential equations, and to point out the application of these developments to the theory of physical systems.

2. *The Law of Uniformity and Ordinary Differential Equations.* Any physical system whose state at any time t is fixed by n real variables $x_1, x_2, \cdots, x_n$ evidently satisfies a set of ordinary differential equations of the form

$$\text{(E)} \qquad \frac{dx_i}{dt} = X_i(x_1, \cdots, x_n), \qquad (i = 1, \cdots, n),$$

which embodies its fundamental law of uniformity. In the case of a Lagrangian physical system, for instance, the $n = 2m$ coordinates x_i are the m geometrical coordinates and their m respective rates of change; in the case of a Hamiltonian system, the coordinates are the m geometrical coordinates and the m corresponding momenta.

I propose to restrict attention to physical systems of the above type E involving a continuous time t and a finite number of degrees of freedom, and thus to forego all consideration of physical systems with a discontinuous time t or an infinite number of degrees of freedom.

3. *The Three Main Types of Physical Systems.* For convenience we shall divide such physical systems into three main types: (a) the general *non-recurrent* systems in which only exceptionally the system recurs to the vicinity of an arbitrary initial state; (b) general *recurrent* systems; (c) *variational* systems derived from a variational principle.

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4. *Example of a Non-Recurrent System.* As a simple example of the non-recurrent type, we shall consider briefly the following.*

Imagine a mass particle which moves in a line subject to an arbitrary force which depends only on position and velocity. If x and t denote the positional coordinate and the time respectively, the differential equation may be written

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right),$$

or in the equivalent form E with $n=2$

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = f(x, y).$$

For definiteness, let us assume that f is analytic and that there is one and only one equilibrium position for the particle, say at the origin $x=0$.

What can be said about the motion of the particle? In answering this question it is convenient to use a geometric representation in the x, y plane. Each motion is represented by a curve $x=f(t)$, $y=dx/dt=f'(t)$, with velocity components $dx/dt=y$ and dy/dt, so that in the upper half plane the point (x, y) moves to the right, in the lower half plane to the left, and along the x-axis vertically.

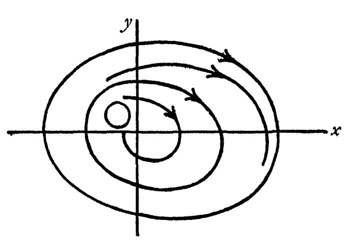

FIG. 1

If now we divide the motions into the stable motions for which x and $dx/dt=y$ remain finite for $t>0$, and the unstable motions, we can readily prove that a stable motion of the particle is periodic, or tends asymptotically towards a periodic motion, or towards the equilibrium position, with expanding or

* See my book on *Dynamical Systems*, 1927, Chap. 5. This book will be referred to hereafter as D.S.

diminishing oscillations. This is true of motions stable in the past as well of those stable in the future.

Similarly, unstable motions oscillate with wider and wider swings in such a way that either the amplitude or the velocity or both become indefinitely large. All this follows from simple considerations based on the analysis situs of the plane of the kind used by Poincaré in his early papers on ordinary differential equations.

It will be seen then that only for the periodic motions is there recurrence. Hence this system is non-recurrent.

Of course in special cases, when there is an energy integral, for instance, all of the motions may be periodic. In such cases the system is recurrent of course.

5. *The Central Motions of Non-Recurrent Systems.* In the general case of non-recurrent systems in n variables, the situation is similar. For definiteness we shall only consider the closed case when $x_1, \cdots, x_n$ may be considered as a point of a closed n-dimensional space M.

This requirement will be realized, for example, if a particle moves on the viscous surface of a sphere subject to an arbitrary force, so that it moves in the direction of the field of force with velocity which is a function of position on the sphere.

Now if we consider all of the points of M, the differential equations E define a steady flow in M. In case there is no tendency towards recurrence, a "molecule" of this fluid will not in general overlap later its initial position. Furthermore, if it does not as time increases, it cannot do so as time decreases. For imagine that the molecule overlaps its first position at time $t=-\tau$. Let time increase by τ. Clearly the position at $t=-\tau$ of the molecule will be restored to the initial position, and the initial position to that at time $t=\tau$, and the overlapping will of course persist. Thus, in the case of non-recurrence, there is a doubly infinite non-overlapping tube formed by the molecule of non-recurring motions. The set of limiting motions M_1 of such non-recurring motions may now be treated in the same way with reference to recurrence and this leads to a subset M_2 of M_1, and so on. Thus we obtain an ordered sequence of sets of motions $M_1, M_2, \cdots$, which terminates in the "central motions" M_c, as I have called them.*

* D.S., Chap. 7.

In the case of the particle on the surface of the sphere, we might have the possibility illustrated below in one hemisphere, in which the limiting great circle together with the equilibrium point E_0 forms the set M_1, and the three equilibrium points E_0, E_1, E_2 form the set M_2 of central motions.

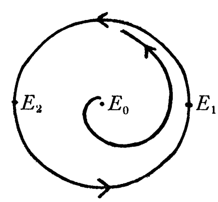

FIG. 2

The two fundamental properties of the central motions are the following: the set of central motions is recurrent in type; the time-probability is 1 that any motion of a physical system is arbitrarily near the set of central motions.

Hence in closed non-recurrent systems the motions will lie (in the sense of time-probability) arbitrarily near the set of central motions of recurrent type.

If the "energy" of a physical system is dissipated indefinitely, it is clear that the corresponding differential equations will have as central motions the equilibrium states of the system.

6. *A Remark Concerning Atomic Systems.* If we consider an atomic system as a dynamical system with n degrees of freedom, at least to a certain degree of approximation, it would seem probable that these equations must be of non-recurrent type and not of the usual recurrent type of classical dynamics. In fact, if the central motions were a set of periodic motions, the system would always tend towards one of these and, when disturbed, would revert to the same periodic motion, or to a different periodic motion if the disturbance were large enough. The relative amount of time spent at any considerable distance from these periodic motions would be very small. Evidently this general situation would agree qualitatively with the facts of radiation.

7. *The Recurrence Theorem of Poincaré.* Suppose next that the corresponding differential equations E are valid in an n-dimensional closed space M, and that the n-dimensional volume of any molecule is not changed as the time changes. Then there

will necessarily be recurrence, as follows from a classical reasoning due to Poincaré.*

In fact consider a small molecule of the fluid at time $t=0$ and at successive intervals $t=\tau, 2\tau, \cdots$. If the volume of the molecule is v, and the total volume of M is V, and if n successive positions do not overlap, we must have $nv \leqq V$. Thus overlapping between some ith and jth position must recur for $i<j\leqq n$ if $n>V/v$. But it follows then that the $(j-i)$th position overlaps the original position; that is, there is recurrence.

By refinement of this mode of argument, Poincaré proved a result which in present-day terminology may be stated as follows. All of the motions corresponding to curves which traverse the given molecule recur to it infinitely often in past as well as in future time, except for a set of measure zero in the sense of Lebesgue, that is, a set which can be enclosed in a numerable set of volumes whose total volume is arbitrarily small.

For a recurrent system all of the motions are central motions; that is, $M=M_1=M_c$. We shall only consider recurrent systems in which such an invariant volume integral exists.

8. *Probability and Physical Systems.* From the standpoint of the physicist it is not the specific motions that are of interest since the initial conditions are not precisely determinable. Nevertheless the mathematician has concerned himself largely with properties of highly improbable special motions such as the periodic motions rather than with the general motions of dynamical systems.

As far as I know it was Koopman, among mathematicians, who first emphasized the importance of getting away from the exclusive consideration of those "properties which are changed altogether by an infinitely small change in the physical conditions attendant on the problem, or by the slightest change in initial data"† and of obtaining results which are valid *in general.*

Evidently results of this type are likely to bring in considerations of *probability.* Indeed Poincaré stated his result in the form

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* *Les Méthodes Nouvelles de la Mécanique Céleste,* vol. 3. See also C. Carathéodory, *Über den Wiederkehrsatz von Poincaré,* Berliner Sitzungsberichte, 1919.

† *Birkhoff on Dynamical Systems,* this Bulletin, vol. 26 (1930), p. 165.

that the probability of recurrence is one under the conditions specified. His theorem is then a step in the desired direction.

9. *The "Ergodic Theorem."* The theoretical physicist has long emphasized the importance of considerations of probability in this field, and on an intuitive basis has formulated vaguely certain types of theorems, one of which in precise form is the following "ergodic theorem" (as I shall call it). *In a closed recurrent system there is a definite time-probability that the moving point not belonging to a certain set of measure zero finds itself in a given region of the space M.*

Very recently von Neumann,* using an important abstract formulation due to Koopman,† of the dynamical problem in terms of linear operations in function space has succeeded in showing that probability considerations can be carried further. His treatment has been given a simplified form by E. Hopf.‡ Von Neumann's result establishes that a similar "mean ergodic theorem" holds in the highly technical sense of "convergence in the mean," although throwing no light on the existence of a time-probability along the individual motions. Nevertheless his result marks a vital step in advance; in fact it is sufficient to solve the statistical problem of classical kinetic theory, provided that the hypothesis of metrical transitivity (§13) be granted, and is of the first order of importance in the realm of ergodic theory.

Shortly thereafter, stimulated by von Neumann's result, I succeeded in proving by entirely new methods that the ergodic theorem holds in the ordinary sense of time probability.§

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* *Proof of the quasi-ergodic hypothesis,* Proceedings of the National Academy of Sciences, January, 1932.

† *Hamiltonian systems and transformations in Hilbert space,* Proceedings of the National Academy of Sciences, May, 1931.

‡ See a note *On the time-average theorem in dynamics,* Proceedings of the National Academy of Sciences, January, 1932. For other recent work in the same direction, see the same journal, E. Hopf, *Complete transitivity and the ergodic principle,* February; B. O. Koopman and J. von Neumann, *Physical applications of the ergodic hypothesis,* March; E. Hopf, *Proof of the Gibbs hypothesis of the tendency towards statistical equilibrium,* April.

§ *Proof of a recurrence theorem for strongly transitive dynamical systems* and *Proof of the ergodic theorem,* Proceedings of the National Academy of Sciences, December, 1931. See also the March issue of the same journal, A. Wintner, *Remarks on the ergodic theorem of Birkhoff.*

A special case of the ergodic theorem is the following: If an $(n-1)$-dimensional open or closed surface σ in M cutting across the curves of motion in one sense be considered, the curves of motion will have a definite mean time of crossing of σ, the same in both directions of time, except for a set of points of Lebesgue measure 0.

This mean crossing-time theorem states then that the following limit exists for every motion except those of a set of measure 0:

$$\lim_{n=\pm\infty} \frac{t_n}{n} = \tau_P.$$

Here t_n denotes the elapsed time to the nth crossing of σ from P on σ. Evidently this is a stronger form of the recurrence theorem of Poincaré.

As a simple application, consider any convex billiard table, and a chalked line l on the table. In general the idealized billiard ball will cross this line with a certain perfectly definite mean time of crossing (which may be infinite). It is known that the billiard ball problem is of recurrent type.

If a certain further condition of metric transitivity (Section 13) is satisfied, this mean time can be easily determined and is the same for all of these motions. On the other hand in special cases (that of an elliptical billiard table for instance) the mean time will not in general be the same for different motions.

10. *Remarks on the Ergodic Theorem.* In order to make plain the nature and scope of the ergodic theorem as applied to recurrent systems, certain remarks need to be made.

In the first place, the theorem applies to physical systems of

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In proving that the limit is the same for $t=-\infty$, my argument in the second note is not properly formulated. However, if the last displayed inequality (p. 659) did not hold, there would be a measurable invariant set $\bar{S}$ for which the stated limit is either greater than μ or is less than λ for $t=-\infty$. Application of the lemma would then give

$$\lambda \int_{\bar{S}} dP \leqq \int_{\bar{S}} t(P)dP \leqq \mu \int_{\bar{S}} dP$$

together with

$$\int_{\bar{S}} t(P)dP > \mu \int_{\bar{S}} dP \text{ or is } < \lambda \int_{\bar{S}} dP$$

which is not possible.

classical dynamical type with given energy constant, in case the n-dimensional space M is closed. For such systems possess the invariant volume integral which necessitates recurrence.

In the second place, when M is open, the theorem applies also to any set of the motions which remain away from the boundary of M, provided that there is such an invariant volume integral.

In the third place, the ergodic theorem may either be expressed in terms of selected volumes in M, or in terms of an arbitrary function $f(P)$ of position on M; in the latter case it states that $f(P)$ will (in general) have a mean value in time along any curve.* The two formulations are at bottom equivalent. It is hardly necessary to state then that the ergodic or time-average theorem permits of innumerable applications.

The exceptional motions left out of account are in general everywhere dense although of measure 0. An analogous case is that of the rational points $x=m/n$ expressed by proper rational fractions. These are everywhere dense on the segment $0<x\leqq 1$ of the line, but are of measure zero, as is well known. For, put an interval of length ϵ^n/n about each such point. There are only $n-1$ points with a given n, so that these are enclosed by intervals of total length less than ϵ^n. Hence all of the rational points are enclosed in a set of total length $\epsilon/(1-\epsilon)$ at most, which can be made arbitrarily small.

In order to bring out the significance of the theorem let us make an application.

Suppose that, in an idealized Sun, Earth and Moon system, these bodies are moving in plane orbits according to Newton's law. Consider the motions of this configuration which are *stable* in the sense that the following inequalities hold for all time:

$$R > r_{SE} > R' > r > r_{EM} > r' > 0,$$

where r_{SE} and r_{EM} designate the distances from the Sun to the Earth and from the Earth to the Moon, respectively. It follows rigorously from the ergodic theorem that, *if* the probability of such stability is not 0, there is a true mean rate of relative rotation of Sun and Moon about the Earth except in an infinitely improbable case.

* See Hopf, loc. cit.

In fact the totality of such motions forms a measurable set. If this set is of measure zero, the probability of stability is 0. Otherwise the ergodic theorem may be applied in the manner indicated to this set of positive measure.

The ergodic theorem tells us nothing about periodicity properties. This may be explained by analogy as follows. If we write down an arbitrary infinite decimal, the limit of the sum of the first n figures divided by n will tend towards $9/2$, the average of $0, 1, \cdots, 9$. This fact tells us nothing about the recurrence of sequences of figures in a particular infinite decimal.

11. *Regional Transitivity.* If now we proceed further with the classification of closed recurrent systems, they may be separated to begin with into the *transitive type*, when an arc of a curve of motion can be found (perhaps very long) which joins a point $\overline{P}$ near an arbitrary point P to a point $\overline{Q}$ near an arbitrary point Q, and the *intransitive type* when this is not true. For transitive systems there exist truly "general motions" which pass infinitely often arbitrarily near all points of M in the future and in the past alike.* Such motions may be called "general motions," in contrast to the "special motions," not having this property.

In the case of such regional transitivity, it is not known, however, whether the general motion is or is not general in the sense of probability.

12. *Example of a Closed System with Regional Transitivity.* I have shown elsewhere that there exist transitive recurrent systems of classical dynamical type with a closed manifold M.† These are formed by the geodesic lines on certain closed two-dimensional surfaces of everywhere negative curvature; such lines are of course the curves of motion of a particle constrained to move in the surface but not acted upon by any force but the force of constraint normal to the surface. The existence of such systems can be rendered intuitively evident in a special case as follows.

The doubly periodic surface

$$z^2 = 1 - e^2 \sin^2 \tfrac{1}{2}x \sin^2 \tfrac{1}{2}y, \qquad (|e| > 1),$$

* D.S., Chap. 7.

† D.S., Chap. 8.

has everywhere negative curvature except at the points for which x and y are both multiples of 2π, where the curvature is zero. Suppose now that we consider the straight lines $x = \pm\pi$ and $y = \pm\pi$ to be joined at corresponding points (conceptually) without distortion of distances on the surface. A surface having the connectivity of the double anchor ring is obtained.

Now any such surface of negative curvature has the property that if an inextensible string be wound on the surface from a point P to Q and pulled taut, it will take a determinate position, namely along the unique geodesic joining P to Q. This geodesic can be characterized topologically by the method of winding the string on the surface, and variation in the position of the two P and Q has very little effect on the intermediate position of the geodesic.* Hence if an infinite string be wound successively in all possible combinations of more and more complicated types, one obtains a geodesic which must approach all possible directions and all possible points. This shows that the system is transitive in the regional sense.

These transitive systems are of Hamiltonian type but a detailed analysis shows that they possess no formally stable periodic motions.

13. *Metrical Transitivity.* The important idea of metrical transitivity may be defined as follows. If every measurable set of curves of motion is either of measure 0 or of the measure V of the volume of M, the system is said to be metrically transitive.

Systems which are metrically transitive are regionally transitive, but the converse is not true. In the case of metrical transitivity the curves of motion are so inextricably intermixed that sets of motion of positive measure cannot be separated off.

The importance of this idea arises from the fact that, almost certainly, recurrent physical systems are *in general* metrically transitive, although this is very difficult to prove.

In the case of metrical transitivity, the ergodic theorem takes the simpler form that the time-probability is given by the ratio v/V of the volume v in question to the volume V of M.

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* The proof of these statements involves reasoning analogous to that used in other connections by Hadamard and Morse; see D.S., Chap. 8.

A very simple example of a metrical transitive system,* with demonstration,† is as follows.

Consider the pair of differential equations

$$\frac{d\phi}{dt} = \frac{\alpha}{2\pi}, \quad \frac{d\psi}{dt} = 1, \qquad (\alpha \text{ irrational}),$$

where ϕ, ψ are angular variables of period 2π. Here the manifold M is a two-dimensional torus, and the stream lines are of the form $2\pi(\phi-\phi_0)=\alpha(\psi-\psi_0)$. A measurable set of curves will correspond to a measurable point set on the "circle" $\psi=0$, given by the points of intersection of these stream lines with $\psi=0$. If we define $f(\phi)$ as 1 or 0 according as the point of this circle with coordinate ϕ does or does not belong to this measurable set, this function f admits a formal Fourier expansion.

$$\cdots + c_{-1}e^{-i\phi} + c_0 + c_1 e^{i\phi} + \cdots, \qquad (i=\sqrt{-1}).$$

But after 2π seconds this point set has moved into itself, each point moving along its curve to a new position $\phi+\alpha$. Thus the series

$$\cdots c_{-1}e^{-\alpha i}e^{-i\phi} + c_0 + c_1 e^{\alpha i}e^{i\phi} + \cdots$$

represents the same function $f(\phi)$. Hence $c_1, c_{-1}, c_2, c_{-2}, \cdots$ are 0 (since α is irrational); and the development of $f(\phi)$ reduces to a constant. But this would mean a measurable set of constant density, that is, either of measure 0 or of measure 2π, as is to be proved.

In all likelihood a proof of metrical transitivity in general will be exceedingly difficult, since the "problem of stability" (Section 17) must first be solved, and this problem has so far defied solution. However in the special system of geodesics on a closed surface of negative curvature, I believe that metrical transitivity can be demonstrated without excessive difficulty, since a complete algorithm exists for the effective treatment of this special type of dynamical system.‡

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* G. D. Birkhoff and P. A. Smith, *Structure analysis of surface transformations*, Journal de Mathématiques, vol. 7 (1928). The definition of metrical transitivity appears in this paper. In this connection, see also P. A. Smith, *The regular components of surface transformations*, American Journal of Mathematics, vol. 52 (1930).

† This proof was discovered independently by Koopman and by Hopf (loc. cit.).

‡ D.S., Chap. 8.

14. *Instability of Periodic Motions in Recurrent Systems.* Before passing on to variational systems it is desirable to point out an important characteristic of the periodic motions of recurrent systems. The linear differential equations of variation along such a motion have periodic coefficients with the period of the motion. There will be in general n particular solutions such that the jth solution is multiplied by a certain constant real or imaginary, ρ_j, when t increases by this period. In the recurrent case the product of these n roots will be 1, but there is no other condition upon the constants ρ_j. Thus there is no reason, except in the case $n=2$, to expect that the characteristic multipliers occur in reciprocal pairs. Hence small perturbations from periodic motion will not in general remain small in either direction of time, since some of the characteristic multipliers will be less and some will be greater than 1 in absolute value.

In fact it may be shown that, unless infinitely many further conditions are satisfied, there will be instability in both directions of time along such periodic motions. As we shall see, this lack of formally stable periodic motions is correlated with a much more rapid circulation of a general point P in the space M than is possible when any periodic motion of formally stable type is present.

15. *Variational Systems.* A string stretched on a smooth surface falls along a geodesic which is the path of a particle moving freely in the surface. This simple fact indicates that the curves of motion of such a particle are obtained from *variational equations*. When such equations are expressed in proper coordinates p_i, q_i, they take the usual Hamiltonian form, with a Hamiltonian function H representing the total energy. The variational principle for these variational systems is then

$$\delta I = \delta \int_{t_0}^{t_1} \{\sum p_i q_i' - H\} dt = 0,$$

where t_0 and t_1 are fixed, and p_i, q_i have given values for $t=t_0$ and $t=t_1$. The curves along which δI vanishes are precisely the solutions of the Hamiltonian equations.

Such sets of equations possess an invariant $2m$-dimensional integral $\int dp_1 \cdots dq_{2m}$. By means of the known energy integral $H=$const., this system may be reduced to one of order $2m-1$ which possesses a corresponding $(2m-1)$-dimensional invariant volume integral. Hence if $H=\text{const.}$ forms a closed space M, the physical system will be recurrent for such a value of the energy constant.

If now we make a *general* transformation from p_i, q_i $(i=1, \cdots, m)$ to $x_i, \cdots, x_{2m}$, the variational principle takes the form

$$\delta I = \delta \int_{t_0}^{t_1} \left\{ \sum X_i x_i' - H \right\} dt = 0,$$

and we obtain a corresponding set of differential equations which I have called Pfaffian.*

This form has the advantage that it is left invariant under a *perfectly arbitrary* transformation of the $2m$ dependent variables, which involves $2m$ arbitrary functions, whereas the Hamiltonian form is only left invariant under certain contact transformations.

It appears highly probable that not only can one pass from the special Hamiltonian form to the Pfaffian, but also that inversely one can reduce the Pfaffian form to the Hamiltonian form. In fact this has already been established by Féraud in certain cases.† If this conjecture be true, it must be regarded as a mere exercise in analytic ingenuity to employ only Hamiltonian systems.

16. *Trigonometric Stability.* In the mathematical treatment of the solar system, it appeared step by step, following Newton, Laplace, and Poisson, that the motion was expressible at least to terms of the second order by means of trigonometric series.

Poincaré was the first, however, to show why, for any Hamiltonian system, it is true that all of the infinitely many conditions for such trigonometric stability are automatically satisfied along any periodic motion, as soon as the usual first-order conditions for stability are fulfilled, namely that the multipliers are distinct imaginary quantities of modulus 1, but are not roots of unity.

For such trigonometrically stable periodic motions, a point P of M distant from the corresponding closed curve by at most ϵ at $t=0$ remains within a distance $L\epsilon$ during a time of order ϵ^{-k}, where k is any arbitrarily large positive integer.

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* D.S., Chaps. 2, 3.

† *On Birkhoff's Pfaffian systems,* Transactions of this Society, vol. 32 (1930); *Extension au cas d'un nombre quelconque de degrés de liberté d'une propriété relative aux systèmes Pfaffiens,* Comptes Rendus, vol. 190 (1930), pp. 358–360.

It may be shown that, from a very broad formal point of view, systems of variational type, like the Hamiltonian and Pfaffian, are the only ones whose periodic motions possess this property.* In consequence of this property of trigonometric stability, the perturbations of motions near stable periodic motions of variational systems can be expressed with an extraordinary degree of accuracy by means of trigonometric series, even if these be actually divergent. Our solar system furnishes an obvious illustration of a physical system of variational type.

From the above point of view the essential significance of the variational systems seems to be this characteristic property of trigonometric stability.

I have also shown (loc. cit.) that variational systems are *reversible in time* from the purely formal point of view, and that, conversely, formally reversible systems will enjoy this property of trigonometric stability. Thus variational character is intimately associated with reversibility, as well as with trigonometric stability.

17. *The Problem of Stability.* In all likelihood trigonometric stability does not mean actual permanent stability. So far, however, the most arduous efforts of mathematicians have failed to show the existence of cases in which a motion arbitrarily near such a stable motion ultimately deviates from it. As long as this possibility is not demonstrated, it will not be possible to prove regional transitivity in any physical system possessing periodic motions of stable type.

However, although ultimate instability remains unestablished, I have recently proved that "rings of instability" at least do exist.†

To explain the significance of this fact let us consider the simplest case of a dynamical system with two degrees of freedom with coordinates p_1, q_1, p_2, q_2. Further let us give attention to a particular energy level, $H=C$.

A periodic motion of stable type is represented by a closed curve C in this three-dimensional space $H=C$. Cut this curve by a two-dimensional element of surface σ at a point Q. Take an

* D.S., Chap. 4.

† *Sur l'existence des régions annulaires d'instabilité,* Annales de l'Institut Henri Poincaré, vol. 2 (1931).

arbitrary point P in the surface and follow along the corresponding curve of motion to the first next point P_1 on σ. We see then that the point P is transformed to P_1, that is, $P_1 = T(P)$. Furthermore, the point Q is invariant: $Q = T(Q)$.

Further investigation shows that the transformation is area-preserving in suitable variables. Conversely any such "conservative" transformation T may be associated with a dynamical system of Hamiltonian type.

In the case when the periodic motion is actually stable, there exists an infinite series of areas invariant under T and closing down upon Q, as was noted by Poincaré (loc. cit.). This means of course that there are tubular regions of complete stream lines in M, which close down upon the curve of periodic motion.

I have shown that the boundaries of such regions form surfaces having a certain amount of smoothness. More precisely, the invariant curves have the form $r = f(\theta)$, where r, θ are polar coordinates and f is a continuous periodic function of period with limited difference quotient. Furthermore these curves form a closed series in a sense which I will not stop to specify.*

A simple possibility is that there exist rings of instability formed by adjacent invariant curves of this description, and this case actually arises, as was stated above. For such rings it may be proved that points arbitrarily near any point of either boundary ultimately will pass arbitrarily near any point of the other boundary. If it could be proved that such a ring can extend to the invariant curve $r = 0$, the problem of stability would be solved, in the sense that trigonometric stability would be shown not to necessitate actual stability.

18. *The Ergodic Function* $T(\epsilon)$. In my opinion, the function $T(\epsilon)$, giving the least time T which elapses before the point P of *some* motion can come within a distance ϵ of every point in M, is destined to play an important part in the characterization of closed transitive physical systems. I will venture therefore to call $T(\epsilon)$ the "ergodic function." For the intransitive recurrent systems there will be an ergodic function for each of the domains of transitivity into which M is divided.

According to the results stated above, in the general varia-

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* *Surface transformations and their dynamical applications*, Acta Mathematica, vol. 43 (1912).

tional case with periodic motions of stable type, a point P cannot leave the ϵ-neighborhood of the corresponding closed curve of motion in time less than ϵ^{-k}. Hence in this case the ergodic function $T(\epsilon)$ increases more rapidly as ϵ diminishes than any negative power ϵ^{-k} of ϵ.

On the other hand, a rough estimate of $T(\epsilon)$ may be attempted in the case of geodesic motion on a closed surface of negative curvature when there are periodic motions of unstable type only. This may be done in the following approximate and non-rigorous fashion.

According to the algorithm referred to, a complete geodesic may be associated with a doubly infinite sequence

$$\cdots a_{-2}a_{-1}a_0a_1a_2 \cdots,$$

where the a's are chosen at pleasure out of a set of N letters

$$\alpha, \beta, \cdots, \nu,$$

each of which represents a fundamental circuit on the surface. Thus the totality of motions is represented by the totality of these symbols. The symbols resemble those of ordinary infinite decimals, except that instead of 10 numerals there are N letters extending to left as well as to right.

If we select one letter as a_0 of this symbol, we are fixing upon that segment of the geodesic which arises from the corresponding circuit.

A finite symbol

$$\cdots [a_{-m} \cdots a_0a_1 \cdots a_m] \cdots$$

in which the letters more than m places distant from a_0 are unspecified will correspond to a three-dimensional volume v in the three-dimensional closed space $M(x, y, \phi)$, representing the points (x, y) together with the directions ϕ corresponding to each state of motion. This volume is obtained from a given segment by continuous variation of the end points as far as possible without alteration of the given finite symbol.

There will be N^{2m+1} of these volumes since there are N^{2m+1} possible finite symbols. These together make up the total volume V of M. It is natural to suppose then that each of these volumes is approximately of the order N^{-2m} of smallness.

Furthermore, these volumes are more or less cylindrical, of length approximately l, where l is the mean length of a fundamental circuit, and so of cross section of order N^{-2m}. It would seem likely, therefore, that the cross-sectional area is of diameter approximately of order N^{-m}. Hence, in order that some particular point P traverse all of M within a distance $\epsilon = N^{-m}$, it is sufficient that the corresponding finite symbol contain all of the possible sequences of $2m+1$ letters of which there are N^{2m+1} in all. The order in which these occur is immaterial, and the number of letters need not be of order higher than N^{2m}.

But the time $T(\epsilon)$ corresponding to the geodesic segment with this finite symbol is of the order of the number of letters. Hence $T(\epsilon)$ is of the order of N^{2m}, that is, of the order of ϵ^{-2} only. This is obviously the minimum possible order.

Thus, in all likelihood, the ergodic function $T(\epsilon)$ *increases only as the* $(n-1)$*st power of the reciprocal of* ϵ *in the general closed recurrent case in n dimensions, whereas it certainly increases more rapidly than any negative power of* ϵ *in the variational case provided a single periodic motion of stable type is present.*

Consequently it is likely that in the general closed recurrent case, an arbitrary motion (aside from those of a set of measure 0) *will traverse all of M within a distance* ϵ *during a time of order* $\epsilon^{-(n-1)}$.

It seems also to be likely that the function $T(\epsilon)$ *increases extraordinarily rapidly towards* ∞ *as* ϵ *tends towards* 0 *in the closed variational case when formally stable periodic motions are present.*

Of course if the problem of stability were solved in the opposite sense to that conjectured above, $T(\epsilon)$ would become infinite for some definite $\epsilon > 0$.

19. *On Open Systems.* Thus far our attention has been directed mainly to closed physical systems. Similar results are valid for physical systems in which the space M of states of motion is open. In fact consider the motions of such a system whose points for $t > 0$ are outside of a certain δ-neighborhood of the boundary of M. These motions may be termed "stable" and form a closed subset of motions of types (a), (b), or (c).

By letting δ approach 0 we obtain the totality of stable motions in the forward sense of time, to which many of the preceding results can be extended.

Thus, in case (a), any stable motion is within an arbitrarily small distance of a central motion nearly always in the sense of time-probability, whereas any unstable motion is within distance ϵ of a central motion or of the singular states of motion corresponding to the boundary of M, nearly always in the same sense.

Even when there is an invariant volume integral a system may be non-recurrent of type (a), if M is open and the total volume is infinite. This happens in the case of the problem of n bodies. Here, in all likelihood, except for a set of motions of measure zero, the n bodies will recede indefinitely from one another either singly or in nearby pairs, and recurrence is impossible.*

Similarly in the recurrent and variational cases (b)† and (c) any stable motion in the forward sense is stable also in the backward sense, save for a set of measure 0. To these stable motions the ergodic theorem applies with slight modification. The remaining unstable motions will be unstable in both directions, save for a set of measure 0. Under certain conditions which we will not state here, the ergodic theorem may be applied to these unstable motions also.

20. *Summary.* Thus in the consideration of the various kinds of physical systems from the standpoint of probability we find three main types.

(a) *Closed non-recurrent systems.* Here the motion tends towards a set of central motions of recurrent type, so that any particular motion is actually within distance ϵ of the central motions nearly always in the sense of time-probability. The simplest possibility is that in which the central motions are equilibrium states or periodic motions.

(b) *Closed recurrent systems.* Here there is recurrence because of the existence of an invariant volume integral over the space M. For the general motion, the "ergodic theorem" ensures general time-average properties of these motions, but does not lead to explicit evaluations. In the case where there is metrical transitivity, these averages are the same for all motions in the sense of probability, and in consequence they can be at once evaluated. The case of metrical transitivity is probably the general case.

* See D.S., Chap. 9.

† The recurrent case is defined as that in which an invariant volume integral exists and there is actual recurrence of any molecule so as to overlap its initial position.

There is instability in both directions of time for all periodic motions in the *general* recurrent case and it seems likely that the time $T(\epsilon)$ (where $T(\epsilon)$ is the "ergodic function") necessary for some motion to come within distance ϵ of all states of motion, will be of approximate order $\epsilon^{-(n-1)}$.

(c) *Closed variational systems.* Here there is recurrence and the ergodic theorem holds as in the general recurrent case. The chief difference between this case and the general recurrent case from the general physical point of view is that $T(\epsilon)$ increases more rapidly than any negative power of ϵ. This is because of the presence in general of periodic motions possessing trigonometric stability, so that motions near such a periodic motion remain nearby during an extremely long interval of time.

Analogous results can be obtained for *open* systems of types (a), (b), or (c) except that it is necessary to deal separately with the unstable motions for which the point of M approaches arbitrarily near the boundary of M.

The outstanding problem concerning physical systems from the point of view of probability is that of determining to what extent recurrent systems are transitive. It is probable that in general there is metrical transitivity. It would be a distinct advance even to establish that there is metrical transitivity in the case of the geodesics on closed surfaces of negative curvature.

The explicit evaluation of the time-averages whose existence is affirmed by the ergodic theorem cannot be made in a given case until the precise nature of the transitivity (or intransitivity) of the motions has been determined.

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