# SETS OF LOCAL SEPARATING POINTS OF A CONTINUUM*

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1. *Introduction*. Let $M$ denote any locally compact metric continuum and let $L$ be the set of all local separating points† of $M$. We proceed to establish the following six properties, of which, for our immediate purposes, the most useful is number (iv).

(i). *If $U$ is any uncountable subset of $L$, there exists a point $x$ of $U$ which is a point of order 2 in $M$ relative to $U$.*

This statement means that $x$ is contained in arbitrarily small neighborhoods whose boundaries have in common with $M$ just two points and these two points belong to $U$. A proof has already been given by the author (loc. cit.).

(ii). *If $H$ is any connected subset of $M$, then $(\overline{H}-H)\cdot L$ is countable.*

For if not, (i) would give a point $x$ of this set which could be separated in $M$ from some point of $H$ by two points not in $H$, which obviously is impossible since $H+x$ is connected.

(iii). *If $H$ is any connected subset of $M$, the points of $H\cdot L$ which are not local separating points of $H$ are countable.*

This results immediately from (i).

(iv). *If $H$ is any connected subset of $M$ such that $\overline{H}\subset L+C$, where $C$ is some countable set, then $H$ is a locally connected $G_\delta$-set. Hence $H$ is arcwise connected.*

By (ii) we see that $(\overline{H}-H)\cdot L$ and hence $\overline{H}-H$ itself is countable. Thus $H$ is a $G_\delta$-set. Now $\overline{H}$ must be a regular curve, for by (i), all save a countable number of its points are points of order 2. Thus any connected subset of $\overline{H}$, and in particular $H$, is locally connected. That $H$ is arcwise connected follows now by the well known theorem of Moore-Menger.‡

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† A point $p$ is a local separating point of $M$ provided some neighborhood $V$ of $p$ exists such that $M\cdot\overline{V}-p$ is separated between some pair of points belonging to the component of $M\cdot\overline{V}$ which contains $p$. See the author's paper in Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 305–314.

‡ See R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of this Society, vol. 13 (1932), p. 86; and K. Menger, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 193–218.

(v). *If $N$ is any continuum $\subset L+C$, where $C$ is countable, then every connected subset of $N$ is a locally connected $G_\delta$-set and hence is arcwise connected.*

This is a corollary to (iv).

(vi). *If $M$ is locally connected, then every connected subset $H$ of $L$ is the difference between an $F_\sigma$ and a countable set.*

For it is known* that in this case $L$ is an $F_\sigma$, so that $M-L$ is a $G_\delta$ and hence so also is $(M-L)\cdot\overline{H}=(M-L)\cdot(\overline{H}-H)$. Whence, $H=[\overline{H}-(M-L)\cdot(\overline{H}-H)]-L\cdot(\overline{H}-H)$, and the first of these two sets is an $F_\sigma$ and the second, by (ii), is countable.

2. THEOREM. *In order that every connected subset of a continuum $M$ be a $G_\delta$ it is necessary and sufficient that the set $N$ of non-local-separating points of $M$ be countable.*

The sufficiency of the condition results immediately from (iv), in view of the fact that, for any connected subset $H$ of $M$, we have $\overline{H}\subset M=L+N$, and $N$ is countable.

To prove that the condition is necessary, we suppose, on the contrary, that $N$ is uncountable. Let $M=M_1+M_2$, where $M_1$ and $M_2$ are disjoint and totally imperfect† and where $M_2$, say, contains uncountably many points of $N$. Then‡ $M_1+L$ is connected and $M-(M_1+L)=E$ is totally imperfect and uncountable. Thus $E$ is not an $F_\sigma$ and hence $M_1+L$ is not a $G_\delta$.

COROLLARY 1. *If every connected subset of $M$ is a $G_\delta$, then $M$ is a regular curve, no cyclic element of $M$ has a continuum of condensation, and the end points of $M$ are countable.*

COROLLARY 2. *If all save a countable number of the points of each cyclic element $C$ of a locally connected continuum $M$ are local separating points of $C$, then every connected subset of $M$ is arcwise connected.*

COROLLARY 3. *If the non-local-separating points of each cyclic element of a locally connected continuum $M$ are countable, then every connected subset of $M$ will be a $G_\delta$ if and only if the end points of $M$ are countable.*

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* See the author's paper, Mathematische Annalen, vol. 162 (1929), p. 318.

† That is, neither contains any perfect set. For a proof that such a division of $M$ is possible, see F. Bernstein, Leipziger Berichte, vol. 60 (1908), p. 325 and Hausdorff, *Mengenlehre*, 1927, p. 156.

‡ See my paper *On the existence of totally imperfect sets* . . . , American Journal of Mathematics, vol. 55 (1933), pp. 146–152.

For any local separating point of a cyclic element of $M$ is a local separating point of $M$ and any non-local-separating point of $M$ which is on no non-degenerate cyclic element of $M$ must be an end point of $M$.

3. THEOREM. *If $M$ is any locally compact continuum such that* (a) *no two maximal free arcs* in $M$ abut and* (b) *$L\cdot\overline{R}$ is countable and $R\neq 0$, where $R$ is the set of all ramification points* (*that is, points of order* $>2$) *of $M$, then $M$ contains a connected subset which is not arcwise connected.*

*Proof.* Let $M=M_1+M_2$, where $M_1$ and $M_2$ are totally imperfect and disjoint. Set $E=M_1+L$. Then since $M-E\subset M_2$, $E$ is connected (loc. cit.). Furthermore $E\cdot\overline{R}=M_1\cdot\overline{R}+L\cdot\overline{R}$, and since $L\cdot\overline{R}$ is countable it follows that $E\cdot\overline{R}$ is totally imperfect. Now if $a$ and $b$ are two points of $E$ lying in different maximal free arcs† of $M$, there can exist no arc $ab$ in $E$. For if $ab$ is any arc in $M$ from $a$ to $n$, then $ab\cdot\overline{R}$ cannot be countable, (for if so, some two free arcs contained in $ab$ would abut), and hence it must contain a perfect set. Thus $ab$ cannot be $\subset E$, since $E\cdot\overline{R}$ is totally imperfect.

4. EXAMPLE. *There exists a regular curve $C$ such that* (a) *no two free arcs of $C$ abut,* (b) *$L\cdot\overline{R}$ is countable, and* (c) *$\overline{R}$ is punctiform. Hence $C$ contains a connected subset which is not arcwise connected.*

Let $I$ be the unit interval and let $K$ be the non-dense perfect set consisting of all numbers on $I$ which can be expressed in the triadic number system using only the digits 0 and 2. Let $I_1$, $I_2, \cdots$, be the segments on $I$ complementary to $K$ ordered in descending order of length and let $p_1, p_2, \cdots$, be the end points of these segments, where $p_{2n-1}$ and $p_{2n}$ are the end points of $I_n$. Let $a_n=\sum_1^n 2^m$ and $P_n=\sum_1^{a_n} p_i$ and $P=\sum_1^\infty P_i$. Let us select in $K-P$ two sequences of points $x_n$ and $y_n$ such that

$$0\leftarrow\cdots<x_n<x_{n-1}<\cdots<x_1<y_1<\cdots<y_{n-1}<y_n<\cdots\rightarrow 1.$$

* An arc $ab$ is *free* in $M$ provided $ab-(a+b)$ is an open subset of $M$. Two such arcs are said to abut if they have a common end point.

† Clearly two such exist, otherwise $M$ contains a punctiform connected set. See my paper, loc. cit.

In the upper half plane let us construct equilateral triangles with bases $x_2y_2$, $y_{4n-1}\,y_{4n+2}$, $x_{4n+2}x_{4n-1}$, $(n=1, 2, 3, \cdots)$, and in the lower half plane construct equilateral triangles with bases $y_{4n-3}y_{4n}$ and $x_{4n}x_{4n-3}$, $(n=1, 2, 3, \cdots)$. Let $T_0$ be the set of triangles so constructed.

Now the points $(x_n)$ and $(y_n)$ together with the points of $P_1$ divide $I$ into a collection of intervals; let us omit $I_1$ from this collection and call $J_1$ the resulting collection. Now on each interval of $J_1$ let us construct a set of equilateral triangles exactly as we constructed $T_0$ on $I$. Let $T_1$ be the set of triangles obtained for all intervals of $J_1$. Now the vertices of the triangles of $T_1$ that are on $I$ together with the points of $P_2$ divide $I$ into a set of intervals. Omit from this set the intervals $I_1$, $I_2$, $I_3$ and call $J_2$ the resulting collection. On each interval of $J_2$ construct a set of triangles as before and call $T_2$ the total set so constructed.

Continue this process indefinitely and let $C=I+\sum_0^\infty T_n$. Then clearly $C$ is a continuum. The maximal free arcs in $C$ are exactly the intervals $I_n$ together with the two sides of each of the triangles of $[T_n]$ not on $I$, so that no two maximal free arcs in $C$ abut. Furthermore the set $R$ of ramification points of $C$ is a subset of $K$ such that $\overline{R}=K$. Thus $\overline{R}$ is totally disconnected and $C$ is a regular curve. Finally we note that if $x$ is any point of $K-(P+R)$, then for each $n$ there is a triangle $t$ of $T_n$ with base $axb$, where $0<a<x<b<1$; and since $\delta(t)<1/(n+1)$ and $x$ is not a ramification point, it follows that $x$ is not a local separating point. Thus $L\cdot\overline{R}=L\cdot K\subset P+L\cdot R$, which shows that $L\cdot\overline{R}$ is countable, since both $P$ and $L\cdot R$ are countable.

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