SOLUTION OF HUNTINGTON'S "UNSOLVED PROBLEM IN BOOLEAN ALGEBRA"

BY A. A. BENNETT

The sixth set of postulates for Boolean algebra recently proposed by E. V. Huntington* may, as he suggests, be modified so as to read as follows. Let

K = an undefined class containing at least two elements, $a, b, c, \cdots$;
T = an undefined subclass in K (so that if a is in T, then a is in K);
$(a+b)$ = the result of an undefined binary operation; and
a' = the result of an undefined unary operation.

POSTULATE 1.71. If a and b are in K, then $(a+b)$ is in K.
POSTULATE 1.7. If a is in K, then a' is in K.
POSTULATE 1.1. If a is in T and $(a'+b)$ is in T, then b is in T.
POSTULATE 1.2. If a is in K, then $[(a+a)'+a]$ is in T.
POSTULATE 1.3. If a, b, etc. are in K, then $[b'+(a+b)]$ is in T.
POSTULATE 1.4. If a, b, etc. are in K, then $[(a+b)'+(b+a)]$ is in T.
POSTULATE 1.6. If a, b, c, etc. are in K, then $\{(b'+c)' + [(a+b)'+(a+c)]\}$ is in T.
POSTULATE 1.8. If $(a'+b)$ is in T and $(b'+a)$ is in T, then $a=b$.

The "unsolved problem" he proposes is the question whether or not Postulate 1.1 is independent of the other postulates in this list. The purpose of the present paper is to answer this question

* E. V. Huntington, *New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica*, Transactions of this Society, vol. 35 (1933), pp. 274–304, especially p. 298. Huntington's sixth set, while inferior to his fourth set when regarded merely as a set of postulates for Boolean algebra, is of interest in connection with B. A. Bernstein's version of the primitive propositions of the *Principia* (see the bibliography in the paper cited). In connection with Huntington's fourth set, it should be noted that Postulate 4.5 is not independent; see the forthcoming number of the Transactions of this Society.

in the affirmative by constructing an actual example of a system $(K, T, +, ')$ which satisfies all the other postulates of the list, but not Postulate 1.1.

For this example, $+$ is nonassociative but is such that I. *If a is in K, then* $(a+a)=a$, and II. *If a and b are in K, then* $(a+b)=(b+a)$. The distinct elements of K are here countably infinite. Each has a unique *rank* equal to the sum of the minimum number of signs, $+$, and $'$, required to represent the element. In particular there is a unique undefined element, e, of rank zero. To obtain all the distinct elements of rank, n, >0, one proceeds as follows by recursion. First, for each previously recorded element a of rank $n-1$, write a'. Secondly, for each element a of rank r, where $(n/2)\leqq r<n$, and for each element, b, of rank $n-r-1$, write $(a+b)$. Thirdly, if n is odd (say $n=2m+1$), for each element a of rank m (other than the last recorded element of rank m) and for each element b of rank m occurring subsequent to a in the recorded list, write $(a+b)$. Thus e' is the only element of rank 1. The distinct elements of rank 2 are e'', $(e'+e)$. Those of rank 3 are e''', $(e'+e)'$, $(e''+e)$, $[(e'+e)+e]$. Similarly for elements of higher rank.

From the method of construction, it is evident that Postulate 1.7 is satisfied, and that for $a\neq b$, Postulate 1.71 is satisfied. By virtue of I, II, this latter postulate holds also for $a=b$.

The subclass T will be defined for this example as consisting exclusively of all elements of K each of which is of one of the four following types, A-D: A, the unique element e; B, $(a'+a)$; C, $[b'+(a+b)]$, for $b\neq a$; D, $\{(b'+c)'+[(a+b)'+(a+c)]\}$, for a, b, c, not all equal. It follows that Postulates 1.2, 1.3, 1.4, 1.6, are satisfied. Indeed Postulate 1.2 reduces by I to B. Postulate 1.3 coincides with C if $b\neq a$, but for $b=a$, reduces by I to B. Postulate 1.4 reduces by II to a special case of B. Postulate 1.6 coincides with D save for $a=b=c$, in which case by use of I it reduces to a special case of B.

From the method of construction of the set of elements of the system one has the following theorem.

III. *If a and b are in K, then $a=b$ if and only if a and b are reducible to identical form in $(e, +, ')$ by at most repeated use of* I *and* II *alone.*

It remains only to show that in this example Postulate 1.8 is satisfied and Postulate 1.1 is violated.

The explicit details required for rigorous proof involve much repetition unless one makes use of lemmas (here denoted by Arabic numerals). While III is frequently invoked, it is needed formally only in certain more specialized forms here listed as Lemmas 1–8, together sufficient to replace III in this discussion. Each of these follows from III by inspection, by the simple expedient of comparing the minimum number of times a sign $+$ or $'$ appears in formal expressions for elements being compared.

1. (i) If $(a+b)=b$, then $a=b$. (ii) If $[a+(b+c)]=c$, then $a=b=c$.

2. If $a\neq b$, and if $c\neq d$, and if $(a+b)=(c+d)$, then either (i) $a=c$ and $b=d$, or else (ii) $a=d$ and $b=c$.

3. a, a', a'', a''', etc., are distinct.

4. If $a'=b'$, then $a=b$.

5. (i) $(a+b')\neq b$, (ii) $(a+b)'\neq b$, (iii) $[(a+b)'+c]\neq b$, (iv) $(a+b')'\neq b$, (v) $[(a+b)'+c]'\neq b$, (vi) $(a+b'')\neq b$.

6. (i) $(u''+u)\neq(a'+a)$, (ii) $(u'''+u)\neq(a'+a)$.

7. If $(a+b)=c'$, then $a=b=c'$.

8. (i) $a'\neq e$, (ii) $(a'+b)\neq e$.

We now prove Lemmas 9–19, using I and II and Lemmas 1–8.

9. (i) $(a+b)\neq b'$, since otherwise by 7, $a=b=b'$, violating 3. (ii) $(a+b)'\neq(a+c)$, since otherwise by 7, $a=(a+b)'$ violating 5(ii), using II. (iii) $(a+b)\neq b'''$, since otherwise by 7, $a=b=b'''$, violating 3.

10. If $c=[(a+b)'+(a+b')]$, then (i) $c\neq a$, (ii) $c\neq b$, (iii) $c'\neq b$, (iv) $c\neq b'$.

Proof. (i) $[(a+b)'+(a+b')]\neq a$ by 5 (iii). (ii) $[(a+b)'+(a+b')]\neq b$, by 5 (iii). (iii) $[(a+b)'+(a+b')]'\neq b$, by 5 (v). (iv) $[(a+b)'+(a+b')]\neq b'$, since otherwise by 7, $(a+b)'=(a+b')$, which violates 9 (ii).

11. $[(a+b)'+(a+c)]\neq d'$, since otherwise by 7, $(a+b)'=(a+c)$, violating 9 (ii).

12. $[(a+b)'+(a+c)]\neq c$, since otherwise by 1 (ii), $(a+b)'=a$, violating 5 (ii).

13. If $\{[(a+b)'+(a+c)]+d\}=(b'+c)$, then $a=b=c$, and hence $d=(a'+a)$.

PROOF. A. Let $c=b'$. Then $\{[(a+b)'+(a+b')]+d\}=b'$, and by 7, $[(a+b)'+(a+b')]=b'$, and again by 7, $(a+b)'=(a+b')=b'$. From $(a+b)'=b'$, by 4 and 1 (i) follows $a=b$.

From $(a+b')=b'$, by 1 (i) or 7 follows $a=b'$. Then $b=b'$ contrary to 3.

B. Let $c\neq b'$, and $d\neq[(a+b)'+(a+c)]$. Then by 2, either (i) $b'=[(a+b)'+(a+c)]$ and $d=c$, or else (ii) $b'=d$, and $c=[(a+b)'+(a+c)]$. But (i) violates 11, and (ii) violates 12.

C. Let $c\neq b'$, but $d=[(a+b)'+(a+c)]$. Then by I, 2 and 4 either (i) $b=(a+b)$ and $c=(a+c)$ or else (ii) $b'=(a+c)$ and $c=(a+b)'$. In (i) by 1(i), $a=b$ and $a=c$. Hence $a=b=c$, as allowed for in the hypothesis of the theorem. In (ii) by 7, $a=c=b'$, and on substituting in $c=(a+b)'$ one has $b'=(b'+b)'$, whence by 4, $b=(b'+b)$, which by 1(i) violates 3.

14. $[(a+b)'+(a+c)]'\neq(b'+c)$, since otherwise by 7, $b'=c=[(a+b)'+(a+c)]'$, and by 4, $b=[(a+b)'+(a+c)]$, violating 5 (iii).

15. (i) $(u'''+u)\neq[b'+(a+b)]$, (ii) $(u'''+u)\neq\{(b'+c)'+[(a+b)'+(a+c)]\}$.

Proof. (i) By 3 and 9 (i) one may apply 2. Under one alternative, $u'''=b'$ and $u=(a+b)$. Hence by 4, one has $u=(a+u'')$ contradicting 5(vi). The other alternative yields $u'''=(a+b)$, $u=b'$. Hence, by 7, $a=b=u'''$, or $u=u^{\text{iv}}$, contradicting 3. (ii) By 3, 11, 2 and 4, either $u''=(b'+c)$ and $u=[(a+b)'+(a+c)]$ or else $u'''=[(a+b)'+(a+c)]$ and $u=(b'+c)'$. In the former case by 7, $b'=c=u''$, and by 4, $b=u'$, whence $[(a+b)'+(a+c)]'=b$ contrary to 5(v). In the latter case by 7, $(a+b)'=(a+c)$ contrary to 9(ii).

16. If $[(x+y)'+(x+z)]=(b'+c)$ and also $[(a+b)'+(a+c)]=(y'+z)$, then $a=b=c=x=y=z$.

Proof. By 11, $c\neq b'$ and $z\neq y'$. By 9(ii), $(x+y)'\neq(x+z)$, and $(a+b)'\neq(a+c)$. Hence, by 2 and 4, one of the four following conditions holds. (i) $(x+y)=b$, $(x+z)=c$, $(a+b)=y$, $(a+c)=z$, or (ii) $(x+y)=b$, $(x+z)=c$, $(a+b)'=z$, $(a+c)=y'$, or (iii) $(x+y)'=c$, $(x+z)=b'$, $(a+b)=y$, $(a+c)=z$, or (iv) $(x+y)'=c$, $(x+z)=b'$, $(a+b)'=z$, $(a+c)=y'$. In (i) we have $[x+(a+b)]=b$, $[x+(a+c)]=c$. Hence by 1(ii), $x=a=b=c$, and upon substituting, $a=b=c=x=y=z$. In (ii), upon substituting one has $[a+(x+y)]'=z$, and $[a+(x+z)]=y'$. Hence by 7, $a=(x+z)=y'$ and hence by 7 again, $a=x=z=y'$. Hence $[y'+(y'+y)]'=y'$, or by 4, $[y'+(y'+y)]=y$ so that by 1(ii), $y=y'$ contradicting 3. Case (iii) differs from case (ii) only by a change of letters throughout. In case (iv) by 7, one has $x=z=b'$, $a=c=y'$.

Hence upon substituting, $(x+y)'=y'$, $(a+b)'=b'$. Hence by 4 and 1(i), $x=y$, $a=b$. But then $b=y'$ and $y=b'$, or $b=b''$ contrary to 3.

17. d' is not in T.

PROOF. A. $d'\neq e$ (by 8(i)); B. $d'\neq(a'+a)$, (by 7 and 3); C. $d'\neq[b'+(a+b)]$ for $a\neq b$, (by 7 and 9(i)); D. $d'\neq\{(b'+c)'+[(a+b)'+(a+c)]\}$, where a, b, c are not all equal, for otherwise by 7, $(b'+c)'=[(a+b)'+(a+c)]$, and by 7 again, $(a+b)'=(a+c)$ contrary to 9(ii).

18. If x and y are distinct elements in K, then $[(x+y)'+y]$ is not in T.

PROOF. A. $[(x+y)'+y]\neq e$, by 8(ii).

B. $[(x+y)'+y]\neq(a'+a)$ for $x\neq y$, since otherwise by 3, 5(ii), 2, and 4, either (i) $(x+y)=a$ and $y=a$, or else (ii) $(x+y)'=a$, and $y=a'$. But in (i) $(x+y)=y$, and hence by 1(i), $x=y$, contrary to hypothesis. In (ii) $(x+a')'=a$, contrary to 5(iv).

C. For $x\neq y$, $[(x+y)'+y]\neq[b'+(a+b)]$, for $a\neq b$. Otherwise by 2 and 4, either (i) $(x+y)=b$ and $y=(a+b)$, or (ii) $(x+y)'=(a+b)$ and $y=b'$. In (i) $[x+(a+b)]=b$, and hence by 1(ii) $x=a=b$, and hence also $x=y$, contrary to hypothesis. In (ii) by 7, $a=b=(x+y)'$, and hence $b=(x+b')'$, contrary to 5(iv).

D. For $x\neq y$, $[(x+y)'+y]\neq\{(b'+c)'+[(a+b)'+(a+c)]\}$, for a, b, c not all equal.

PROOF. By 11, $[(a+b)'+(a+c)]\neq(b'+c)'$, and by 5(i), $y\neq(x+y)'$. Hence were the theorem false in this case, by 2 and 4, either (i) $(x+y)=(b'+c)$ and $y=[(a+b)'+(a+c)]$, or else (ii) $(x+y)'=[(a+b)'+(a+c)]$ and $y=(b'+c)'$. In (i) $\{x+[(a+b)'+(a+c)]\}=(b'+c)$. Hence by 13, $a=b=c$, and $x=(a'+a)=y$, contrary to hypothesis. In (ii) by 7, $(a+b)'=(a+c)$ contrary to 9(ii).

19. $\{[(x+y)'+(x+z)]'+(y'+z)\}$ is not in T if x, y, z are not all equal.

PROOF. A. $\{[(x+y)'+(x+z)]'+(y'+z)\}\neq e$, by 8(ii).

B. $\{[(x+y)'+(x+z)]'+(y'+z)\}\neq(a'+a)$, unless $x=y=z$. By 14, 3, 2, and 4, either (i) $[(x+y)'+(x+z)]=a$ and $(y'+z)=a$, or (ii) $[(x+y)'+(x+z)]'=a$, and $(y'+z)=a'$. In (i), $[(x+y)'+(x+z)]=(y'+z)$ which by 13 (for $d=[(x+y)'+(x+z)]$) is only possible for $x=y=z$. In (ii), $[(x+y)'+(x+z)]''=(y'+z)$.

Hence by 7, $z=y'=[(x+y)'+(x+z)]''$ and by 4, $y=[(x+y)'+(x+z)]'$, contradicting 5(v).

C. For x, y, z not all equal, $\{[(x+y)'+(x+z)]'+(y'+z)\} \neq [b'+(a+b)]$, where $a \neq b$. By 14, 9(i), 2 and 4, either (i) $[(x+y)'+(x+z)]=b$ and $(y'+z)=(a+b)$, or e'se (ii) $[(x+y)'+(x+z)]'=(a+b)$ and $(y'+z)=b'$. In (i) $(y'+z)=\{a+[(x+y)'+(x+z)]\}$ which by 13 is possible only for $x=y=z$. In (ii), by 7 and 4, $a=b=[(x+y)'+(x+z)]'$, $y=b$, $z=y'$. Hence upon substituting, $[(x+y)'+(x+y')]'=y$ contradicting 10(iii).

D. For x, y, z not all equal, $\{[(x+y)'+(x+z)]'+(y'+z)\} \neq \{(b'+c)'+[(a+b)'+(a+c)]\}$, where a, b, c, are not all equal. Otherwise by 14, 11, 2 and 4, either (i) $[(x+y)'+(x+z)]=(b'+c)$ and $(y'+z)=[(a+b)'+(a+c)]$, or else (ii) $[(x+y)'+(x+z)]'=[(a+b)'+(a+c)]$ and $(y'+z)=(b'+c)'$. In case (i) by 16, $a=b=c=x=y=z$, excluded by hypothesis. In case (ii) by 7, $(a+b)'=(a+c)$ contrary to 9(ii).

THEOREM. *Postulate* 1.8 *is satisfied.*

PROOF. The postulate may be restated as follows: *If $u \neq v$, and if $(u'+v)$ is in T, then $(v'+u)$ is not in T.* To prove this one need only test the possible expressions for $(u'+v)$, with $u \neq v$, which are in T. One may note first that $v \neq u'$, since by 17, u' is not in T. Consider in turn the available alternatives A–D. A. $(u'+v) \neq e$, by 8(ii). B. Let $(u'+v)=(a'+a)$. Since $v \neq u'$, one has by 2 and 4, either $u=a=v$, (contrary to hypothesis), or else $u'=a$ and $v=a'$, so that $(v'+u)=(u'''+u)$. But the possible alternatives A–D for this subcase are to be successively rejected by use of 8(ii), 6(ii), 15(i) and (ii). C. Let $(u'+v)=[b'+(a+b)]$, for $a \neq b$. Then by 2 and 4, either (i) $u=b$, and $v=(a+b)$ or (ii) $u'=(a+b)$ and $v=b'$. In (i) $(v'+u)=[(a+b)'+b]$, with $a \neq b$. But by 18, this is not in T. In (ii), by 7, $a=b$, contrary to hypothesis. D. Let $(u'+v)=\{(b'+c)'+[(a+b)'+(a+c)]\}$, with $u \neq v$, and with a, b, c not all equal. Then in view of 17, by 2 and 4, it follows that either (i) $u=(b'+c)$ and $v=[(a+b)'+(a+c)]$ or else (ii) $u'=[(a+b)'+(a+c)]$ and $v=(b'+c)'$. In (i) $(v'+u)$ cannot be in T by 19 while (ii) contradicts 11.

THEOREM. *Postulate* 1.1 *fails to hold.*

PROOF. e is in T, (A), and hence, (C), $[e'+(e''+e)]$ is in T. But as is now to be shown $(e''+e)$ is not in T. Indeed, we have

A. $(e''+e)\neq e$, by 8(ii). B. $(e''+e)\neq(a'+a)$, by 6(i). C. $(e''+e)\neq[b'+(a+b)]$. For otherwise, by 3, 9(i), 2 and 4, either (i) $e'=b$ and $e=(a+b)$, or else (ii) $e=b'$ and $e''=(a+b)$. But (i) is impossible since $(a+b)'\neq b$ by 5(ii), and (ii) is impossible since $e\neq b'$ by 8(i). D. $(e''+e)\neq\{(b'+c)'+[(a+b)'+(a+c)]\}$. Indeed otherwise in view of 3, 11, 2 and 4, either (i) $e'=(b'+c)$ and $e=[(a+b)'+(a+c)]$ which contradicts 8(ii), or else (ii) $e''=[(a+b)'+(a+c)]$ and $e=(b'+c)'$ which contradicts 8(i) and also 11.

Brown University

CONCURRENCE AND UNCOUNTABILITY*

BY N. E. RUTT

1. *Introduction.* The point set of chief interest in this paper, a plane bounded continuum Z, is the sum of a continuum X and a class of connected sets $[X_\alpha]$, each element X_a of which has at least one limit point in X and is a closed subset of $c_u(X+X_b)$, where X_b is any element of $[X_\alpha]$ different from X_a and where $c_u(X+X_b)$ is the unbounded component of the plane complement of the set $X+X_b$. Upon a basis of separation properties, order† may be assigned to the elements of $[X_\alpha]$ agreeing in its details with that of some subset of a simple closed curve. We shall use some definite element X_r of $[X_\alpha]$ as reference element, selecting as X_r one of $[X_\alpha]$ containing a point arcwise accessible from $c_u(Z)$. A countable subcollection $[X_i^h]$ of $[X_\alpha]$ excluding X_r is called a *series* if for each j, $(j=2, 3, 4, \cdots)$, the elements X_j and X_r separate X_{j-1} and X_{j+1}. Two different series $[X_i^h]$ and $[X_i^k]$ are said to be *opposite in sense* if there exist different subscripts m and n such that X_m^h and X_m^k separate both X_n^h and X_n^k from X_r; otherwise they are said to have the *same sense.* They are said to be *concurrent* if they have the same sense and if there exists no element of $[X_\alpha]$ which together

* Presented to the Society, February 25, 1933.

† R. L. Moore, *Concerning the sum of a countable number of continua in the plane*, Fundamenta Mathematicae, vol. 6, pp. 189–202; J. H. Roberts, *Concerning collections of continua not all bounded*, American Journal of Mathematics, vol. 52 (1930), pp. 551–562; N. E. Rutt, *On certain types of plane continua*, Transactions of this Society, vol. 33, No. 3, pp. 806–816.