ON THE NUMBER OF STATIONARY TANGENT S_{t-1}'S TO A V_k^n IN AN S_{tk+k-1}

BY B. C. WONG

In this paper we propose to determine, geometrically, the number N of stationary tangent S_{t-1}'s to a k-dimensional variety V_k^n of order n in an S_{tk+k-1}. As the general problem of determining N for a V_k^n which is the locus of ∞^h S_{k-h}'s for $h>1$ offers some difficulty, we shall confine ourselves to the case where V_k^n is a non-developable locus of ∞^1 S_{k-1}'s. By a stationary tangent S_{t-1} to V_k^n we mean an S_{t-1} meeting V_k^n in $t+1$ consecutively coincident points, that is, meeting $t+1$ consecutive S_{k-1}'s of V_k^n each in a point.

Suppose that V_k^n belongs to an S_r. Now in S_r there are $\infty^{t(r+1-t)}$ S_{t-1}'s. For an S_{t-1} to meet V_k^n in $t+1$ points is equivalent to $(t+1, (r+1-k-t)$ conditions; and if the $t+1$ points of intersection are to be consecutively coincident, t further conditions will be absorbed. In order that the number N of stationary tangent S_{t-1}'s to V_k^n be finite, the dimension r of the space containing V_k^n must be such that $(t+1)(r+1-k-t)+t=t(r+1-t)$, from which we obtain $r=tk+k-1$. Our problem is to find N for V_k^n in S_{tk+k-1}.

Here we find it necessary to give two known results of which we shall make use subsequently.

(I) Let there be given q varieties $V_{x_1}^{m_1}, V_{x_2}^{m_2}, \cdots, V_{x_q}^{m_q}$ of orders $m_1, m_2, \cdots, m_q$, respectively, such that $V_{x_i}^{m_i}$ is the locus of ∞^1 (x_i-1)-spaces. If there exists a one-to-one correspondence between the elements of these varieties, then the locus of the ∞^1 $(x_1+x_2+\cdots+x_q-1)$-spaces determined by corresponding elements is a $V_{x_1+x_2+\cdots+x_q}$ of order $m_1+m_2+\cdots+m_q$.*

(II) The locus of the ∞^1 S_x's each meeting in $x+1$ coincident points a given curve of order m and deficiency p is a developable V_{x+1}^M of order $M=(x+1)(m-x+xp)$.†

* This result is a generalization of the proposition that the locus of the lines joining corresponding points of two projectively related curves is a ruled surface of order equal to the sum of the orders of the two curves.

† Veronese, *Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens*, Mathematische Annalen, vol. 19 (1882), pp. 161–234.

Now we proceed to the determination of N for V_k^n. We regard V_k^n as the projection of a $V_k'^n$ of the same order in an S_{tk+k+1} containing S_{tk+k-1}. Let there be given in S_{tk+k+1} k curves C^{m_1}, $C^{m_2}, \cdots, C^{m_k}$ of orders $m_1, m_2, \cdots, m_k$, respectively, and all of the same deficiency p, and let a one-to-one correspondence exist between the points of these curves. Each group of corresponding points determines an S'_{k-1} and the locus of the ∞^1 S'_{k-1}'s so determined is, according to (I), a $V_k'^n$ of order $n=m_1+m_2+\cdots+m_k$. These S'_{k-1}'s will be called generating S'_{k-1}'s of $V_k'^n$ and the given curves will be called directrix curves, that is, curves each meeting each generating S'_{k-1} in a point.

Consider $t+1$ consecutive generating S'_{k-1}'s of $V_k'^n$. They determine a $(tk+k-1)$-space R_{tk+k-1}. The locus of the ∞^1 R_{tk+k-1}'s so determined is a V_{tk+k}^{ν} of order ν. To find ν, notice that an R_{tk+k-1} contains $t+1$ consecutive points on each of the k directrix curves and therefore the S_t determined by these $t+1$ points. The developable $V_{t+1}^{\mu_i}$ to the curve C^{m_i} is, according to (II), of order $\mu_i=(t+1)(m_i-t+tp)$. Then, from (I), the order of the locus V_{tk+k}^{ν} is $\nu=(t+1)(n-tk+tkp)$.

Now project $V_k'^n$ upon an S_{tk+k-1} from a general line l of S_{tk+k+1} and let the projection be V_k^n. The line l meets the locus V_{tk+k}^{ν} in ν points, that is, meets ν of the R_{tk+k-1}'s of V_{tk+k}^{ν}. Consider one of these R_{tk+k-1}'s and let the point of its incidence with l be P. From P one and only one S_t can be constructed meeting the $t+1$ consecutive S'_{k-1}'s of $V_k'^n$ lying in R_{tk+k-1} each in a point. Now an S_{t+1} containing l and S_t meets the S_{tk+k-1} upon which $V_k'^n$ is being projected in an S_{t-1}. This S_{t-1}, intersecting the projection V_k^n in $t+1$ consecutively coincident points which are the projections of the points on $t+1$ consecutive S'_{k-1}'s of $V_k'^n$, is a stationary tangent S_{t-1} of V_k^n. The number N of such stationary tangent S_{t-1}'s is evidently equal to the number of points in which l meets the locus V_{tk+k}^{ν} in S_{tk+k+1}, that is,

$$N = \nu = (t+1)(n-tk+tkp),$$

which was to be found.

We give a few illustrations of this formula. For $k=1$, we have a curve C^n in S_t, and the number of stationary tangent S_{t-1}'s to C^n is $N=(t+1)(n-t+tp)$. This particular result can be at once derived from (II) by projecting the curve upon a space of t dimensions. Now let $k=2$. If $t=1$, it follows that

$N=2(n-2+2p)$ is the number of pinch points on a ruled surface in S_3 and if $t=2$, then $N=3(n-4+4p)$ is the number of inflexional tangent lines to a ruled surface in S_5. We also see that a V_3^n, given by $k=3$, has $2(n-3+3p)$ pinch points if it is in S_6 and $3(n-6+6p)$ inflexional tangent lines if it is in S_8; and so forth.

THE UNIVERSITY OF CALIFORNIA

CLASS NUMBER IN A LINEAR ASSOCIATIVE ALGEBRA*

BY GRACE SHOVER

1. *Introduction*. In this paper the finiteness of the class number is established for every division algebra taken over the rational field. For every semi-simple algebra, the right and left class numbers are proved to be equal. The classical method for proving the finiteness of the class number in algebraic fields depends upon the multiplication of ideals, but the problem is treated in this paper for the general case without reference to the concept of ideal multiplication. The finiteness of the class number for every algebraic field follows as a special case.

2. *Definitions and References*. Algebra, domain of integrity, and ideal are defined as in a previous paper.†

The norm of an ideal $\mathfrak{R}$ is defined to be the absolute value of $|G|$, where G is a matrix representing $\mathfrak{R}$.‡

A necessary and sufficient condition that a matrix G represent a left (right) ideal is that

$$GR_p{}^{\mathsf{T}} = D_pG, \ (GS_p = Q_pG), \quad (p = 1, 2, \cdots, n),$$

where $R_p{}^{\mathsf{T}}$ is the transpose of the first matrix of e_p (S_p is the second matrix of e_p). The matrices $D_p(Q_p)$ are called the class matrices corresponding to the ideal matrix G.

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† Shover and MacDuffee, this Bulletin, vol. 37 (1931), pp. 434–438.

‡ MacDuffee, Transactions of this Society, vol. 31 (1929), pp. 71–90.