NORMAL DIVISION ALGEBRAS OVER ALGEBRAIC NUMBER FIELDS NOT OF FINITE DEGREE*

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1. *Introduction.* If R is the field of all rational numbers and if $\xi_1, \cdots, \xi_n$ are ordinary algebraic numbers, then the field $\Omega = R(\xi_1, \cdots, \xi_n)$ of all rational functions with rational coefficients of $\xi_1, \cdots, \xi_n$ is an *algebraic number field of finite degree* (the maximum number of linearly independent quantities of Ω) *over* R. It has recently been proved† that every normal *simple* algebra over such a field Ω is cyclic. In particular it has been shown that every normal *division* algebra of order n^2 (degree n) over Ω is cyclic and has exponent n.

In the present note I shall give an extension of the above results to *normal division algebras over any algebraic number field* Λ. I shall prove that all normal division algebras over Λ are cyclic and with degree equal to exponent but shall give a trivial example showing that the theorem corresponding to the above on normal *simple* algebras is false. The problem of the equivalence of normal division algebras over Λ will also be discussed.

2. *Cyclic Algebras.* Let F be any non-modular field and let Z by cyclic of degree n over F. Then Z possesses a generating automorphism

$$S: \qquad z \longleftrightarrow z^S, \qquad (z \text{ in } Z,\ z^S \text{ in } Z),$$

such that every automorphism of Z is one of $S^0 = S^n = I$, S, $S^2, \cdots, S^{n-1}$. The algebra A of all quantities

$$\sum_{i=0}^{n-1} z_i y^i, \qquad (z \text{ in } Z),$$

is a cyclic algebra with multiplication table

$$y^n = \gamma \text{ in } F,\ y^e z = z^{S^e} y^e, \qquad (e = 0, 1, \cdots),$$

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† See the paper by H. Hasse and myself in the Transactions of this Society, vol. 34 (1932), pp. 722–726, for the normal division algebra theorem. The theorem for normal simple algebras follows from Hasse's Theorem 6 of his Transactions paper, vol. 34 (1932), pp. 171–214.

for every z of Z. Evidently A is uniquely defined by Z, S, γ, and thus we write

$$A = (Z, S, \gamma).$$

Let F be contained in any larger field K. Then

$$A_K = (Z, S, \gamma)_K$$

is the algebra with the same basis and constants of multiplication as A, but over K.

If A_K is a division algebra, then so evidently is A. But then Z_K, which is the algebra with the same basis and constants of multiplication as the field Z, but over K, is a field and in fact is evidently cyclic of degree n over K. Evidently $A = (Z_K, S, \gamma)$ over K.

THEOREM 1. *Let $A = (Z, S, \gamma)$ over F, $F < K$, and let A_K be a division algebra. Then A_K is the cyclic algebra (Z_K, S, γ) over K.*

3. *The Determination of Algebras over* Λ. Let Λ be any non-modular field whose quantities are all algebraic numbers and let A be a normal division algebra of order $m = n^2$ over Λ. If $u_1, \cdots, u_m$ are a basis of A, then $u_i u_j = \sum' \gamma_{ijk} u_k$ with γ_{ijk} in Λ. But then γ_{ijk} are all algebraic numbers, so that $L = R(\gamma_{111}, \cdots, \gamma_{ijk}, \cdots, \gamma_{mmm})$ is algebraic of finite degree.

The linear set $B = (u_1, \cdots, u_m)$ over L is evidently an algebra of order m over L. If in particular $u_1 = 1$, the modulus of A, then u_1 is the modulus of B. Evidently $A = B_\Lambda$.

If B contains any divisors of zero, then these quantities are in the division algebra A, a contradiction. Hence B is a division algebra.

Let B contain a quantity $k = \sum \lambda_i u_i$, λ_i in L, which is commutative with every quantity of B. In particular $k u_i = u_i k$, so that $k(\sum \mu_i u_i) = (\sum \mu_i u_i) k$ for μ_i any quantities of the field Λ. But A is normal, so that k is a multiple of the modulus u_1 of A by a quantity of Λ. Hence $k = \mu u_1 = \sum \lambda_i u_i$. Since the u_i are linearly independent in Λ, we have $\mu = \lambda_1$, k is a multiple of u_1 by a quantity of L, and B is normal.

The normal division algebra B of degree n over L is thus* a cyclic algebra (Z, S, γ) over L. The basis, (u_i), of A is linearly

* By the result already quoted on normal division algebras over Ω.

expressible with coefficients in L in terms of the basis of $B=(Z, S, \gamma)$ in its cyclic form, so that in fact $A=(Z, S, \gamma)_\Lambda$. By Theorem 1 we have the following result.

THEOREM 2. *Let A be a normal division algebra of degree n over an algebraic number field Λ not of finite degree. Then there exists a sub-field L (of Λ) of finite degree and a cyclic algebra, $B=(Z, S, \gamma)$, over L such that $A=(Z_\Lambda, S, \gamma)$ over Λ, where Z_Λ is a cyclic field of degree n over Λ. Hence A is cyclic.*

4. *The Exponent of Algebras A.* Suppose that the algebra A of Theorem 2 has exponent $\rho < n$. Then A^ρ is well known to be equal to $M^{\rho-1}\times(Z_\Lambda, S, \gamma^\rho)$, where M is a total matric algebra. But A^ρ is a total matric algebra; hence $(Z_\Lambda, S, \gamma^\rho)$ is also. Hence γ^ρ is the norm $N(c)$ of a quantity c of Z_Λ.

Let $Z=L(x)$, $Z_\Lambda=\Lambda(x)$, so that $c=\sum c_i x^i$, where the c_i are in Λ. The field $L=L(c_0, \cdots, c_{n-1})$ is algebraic of finite degree. Moreover, if $B=(Z, S, \gamma)$, then evidently $Z_0=L_0(x)$, $B_0=(Z_0, S, \gamma)$ over L_0, is contained in A and hence is a cyclic division algebra. But $B_0^\rho=(Z_0, S, \gamma^\rho)\times M^{\rho-1}$ is a total matric algebra, since $\gamma^\rho=N(c)$, where c is in Z_0.

The exponent of B_0 of degree n over L_0 is known to be n since B is a cyclic division algebra over L_0, which is *algebraic of finite degree.* Hence $\rho \geqq n$, a contradiction.

THEOREM 3. *The exponent of any normal division algebra over Λ is its degree.*

5. *On the Equivalence of Algebras over* Λ. Let $A=(Z_\Lambda, S, \gamma)$ and $C=(Y_\Lambda, T, \delta)$ over Λ be normal division algebras. Then Z and γ are obtained with respect to a field L_1 defined by A, Y, and δ with respect to L_2 defined by C. If L is the composite of L_1 and L_2, then we may evidently take L as the common field of Theorem 2 for both algebras A and C. Hence $A=(Z, S, \gamma)_\Lambda$, (Z, S, γ) a normal division algebra over L, $C=(Y, T, \delta)_\Lambda$, (Y, T, δ) also a normal division algebra over L.

The algebra A is equivalent to the algebra C if and only if $A\times C^{-1}=(Z, S, \gamma)\times(Y, T, \delta^{-1})$ is a total matric algebra. But, as is well known, $(Z, S, \gamma)\times(Y, T, \delta^{-1})=(X, R, \epsilon)\times M$, where M is a total matric algebra and (X, R, ϵ) is a uniquely determined cyclic algebra. Evidently $A\times C^{-1}$ is total matric if and only if $(X, R, \epsilon)_\Lambda$ is total matric. For $A\times C^{-1}=M\times(X, R, \epsilon)_\Lambda$.

But then $\epsilon = N(c)$, where c is in X_Λ. As before there exists a sub-field L_0 of Λ of finite degree such that c is in X_{L_0}, $(X, R, \epsilon)_{L_0}$ is total matric. But then $(Z, S, \gamma)_{L_0}$ is equivalent to $(Y, T, \delta)_{L_0}$. The converse is obvious and we have proved this theorem.

THEOREM 4. *Let A and C be normal division algebras of degree n over Λ, an algebraic field not of finite degree, so that $A = (Z_\Lambda, S, \gamma)$, $C = (Y_\Lambda, T, \delta)$, where $B = (Z, S, \gamma)$, $D = (Y, T, \delta)$ are cyclic over the same sub-field L of finite degree of Λ. Then A and C are equivalent if and only if there exists a sub-field $L_0 > L$ of Λ such that L_0 has finite degree and the algebras B_{L_0} and D_{L_0} are equivalent.*

The above theorem essentially reduces the problem of the equivalence of normal division algebras over A to the corresponding problem (already solved*) for algebras over fields of finite degree, and to a consideration of the sub-fields of Λ of finite degree.

6. *Normal Simple Algebras over* Λ. In this section we shall show trivially that there exist non-cyclic normal simple algebras over an algebraic field Λ. We take Λ to be the field of all constructible (with ruler and compass) numbers, extended by $i = (-1)^{1/2}$. That is, we take Λ to consist of all numbers obtained from rational numbers by a finite number of rational operations and extractions of square roots.

Evidently any equation $x^2 = c$, c in Λ, is reducible in Λ since $c^{1/2}$ is also in Λ. But then there exist no cyclic algebras of degree *two* over Λ. Hence the total matric algebra of degree two over Λ, a normal simple algebra, is non-cyclic.

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* See Hasse, loc. cit.