ON THE NUMBER OF APPARENT DOUBLE POINTS ON A CERTAIN V_k^n IN AN S_{2k+1}

BY B. C. WONG

Consider a k-dimensional variety, V_k^n, of order n which is the locus of a single infinity of $(k-1)$-spaces in an S_{2k+1}. It is known that such a V_k^n, if it is rational, that is, if its section by a general S_{k+2} of S_{2k+1} is a rational curve, has*

$$b_k = \tfrac{1}{2}(n - k)(n - k - 1)$$

apparent double points.† The question arises: What is the value of b_k when V_k^n is not rational? The case $k=1$ is familiar; a curve of order n in an S_3 has

$$b_1 = \tfrac{1}{2}(n - 1)(n - 2) - p \tag{1}$$

apparent double points, where p is the deficiency of the curve. It is also known that, for $k=2$, the number of apparent double points on a ruled surface F^n of order n in an S_5 is‡

$$b_2 = \tfrac{1}{2}(n - 2)(n - 3) - 3p, \tag{2}$$

where p is the deficiency of the curve of intersection of F^n by a general S_4 of S_5. For $k>2$, the number b_k of apparent double points of a V_k^n in an S_{2k+1} seems to be as yet unknown. It is our purpose in this note to derive a formula for this number.

Now let V_k^n be intersected by a general S_{k+2} of S_{2k+1} in a curve C^n of deficiency p. If $p>0$, we say that V_k^n is not rational. We shall say that p is also the deficiency of V_k^n and shall regard n and p as the two essential characteristics of the variety as all its other characteristics can be expressed in terms of them for a

---

* B. C. Wong, *On the number of* $(q+1)$-*secant* S_{q-1}'*s of a certain* V_k^n *in an* $S_{qk+q+k+1}$, this Bulletin, vol. 39, pp. 392–394.

† By an apparent double point of a V_k^n we mean a secant line of V_k^n passing through a given point of S_{2k+1}. The projection in an S_{2k} of V_k^n will have b_k improper double points each of which is the projection of an apparent double point of V_k^n.

‡ Severi, *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a'suoi punti tripli apparenti*, Rendiconti di Palermo, vol. 15 (1901), pp. 33–51.

given value of k. Then the formula for b_k must be a function of n and p, and of k also.

Consider the ruled surface F^n in which V_k^n is met by a general S_{k+3} of S_{2k+1}. The projection of F^n, if it is in an S_5, has b_2, given by formula (2), apparent double points; and, if it is in an S_3, has a double curve of order b_1 given by formula (1). On this double curve lie a finite number, j_1, of pinch points. This number is known and will be given subsequently.

Next, consider the planed variety V_3^n common to V_k^n and a general S_{k+4} of S_{2k+1}. The projection in an S_7 of this V_3^n has b_3 apparent double points. Projecting this projection successively upon an S_6, an S_5, and an S_4, we see that the resulting variety in S_6 has b_3 improper double points; that the one in S_5 contains a double curve of order b_2 upon which lie j_2 pinch points; and, finally, that the one in S_4 contains a double surface of order b_1 upon which lies a pinch curve of order j_1.

In general, an $S_{k+h+1}(k \geqq h > 0)$ of S_{2k+1} meets V_k^n in a V_h^n which is the locus of a single infinity of $(h-1)$-spaces. Now if we let V_h^n be projected upon an S_{2h-i}, $(i=0, 1, \cdots, h-1)$, of S_{k+h+1}, then we have for projection an h-dimensional variety of order n with a double i-dimensional variety of order b_{h-i} and an $(i-1)$-dimensional pinch variety of order j_{h-1} lying on the double variety. If $i=0$, the projection in S_{2h} has b_h improper double points.

Suppose $h=k$, and then we have the given V_k^n itself. A general S_{2k-i}-projection of this V_k^n contains a double i-dimensional variety of order b_{k-i} upon which lies an $(i-1)$-dimensional variety of order j_{k-i}.

In order to determine b_k we find it necessary to determine b_h. This determination will be much facilitated if we make use of the two following results already known.

(A) The characteristics $b_0, b_1, \cdots, b_k; j_0, j_1, \cdots, j_{k-1}$ of any V_k^n in r-spaces satisfy the relations*

$$\begin{aligned} 2b_h &= 2b_{h-1} - j_{h-1} \\ &= 2b_{h-2} - j_{h-1} - j_{h-2} \\ &\quad \cdots\cdots\cdots\cdots \\ &\quad \cdots\cdots\cdots\cdots \end{aligned}$$

* B. C. Wong, *On certain characteristics of k-dimensional varieties in r-space*, this Bulletin, vol. 38, pp. 725–730. The notations used in this paper are slightly different from those adopted in the present work.

$$= 2b_1 - j_{h-1} - j_{h-2} - \cdots - j_1$$
$$= 2b_0 - j_{h-1} - j_{h-2} - \cdots - j_1 - j_0.$$

Here b_0 is to be taken equal to $n(n-1)/2$* and j_0 is the rank of the curve C^n common to V_k^n and an S_{k+2}.†

(B) The number of pinch points on the double curve of a V_h^n which is the locus of a single infinity of S_{h-1}'s in an S_{2h-1} is‡

$$j_{h-1} = 2(n - h + hp).$$

Combining these two results, we find that

$$b_h = b_0 - (1/2) \sum_{i=0}^{h-1} j_i$$
$$= (n - h)(n - h - 1)/2 - k(k + 1)p/2.$$

If $h=1$ and 2, we have formulas (1) and (2), respectively. For $h=k$, we have

$$b_k = (n - k)(n - k - 1)/2 - k(k + 1)p/2$$

as the number of apparent double points on a V_k^n which is the locus of a single infinity of S_{k-1}'s in an S_{2k+1} and this is the number it was our purpose to determine.

THE UNIVERSITY OF CALIFORNIA

---

* If we define b_h as the number of secant lines of a V_h^n of an S_{2h+2} that meet a given line of S_{2h+2}, we see that b_0 is the number of lines determined by n given points in a plane.

† We may define j_{h-1} as the number of tangent lines of a V_h^n of an S_{2h} that pass through a given point of S_{2h}. Then, j_0 is the class of a plane curve which is the plane projection of the curve C^n of intersection of V_k^n and an S_{k+2}.

‡ B. C. Wong, *On the number of stationary tangent* S_{t-1}*'s to a* V_k *in an* S_{tk+k-1}, this Bulletin, vol. 39, pp. 608–610.