ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. Professor D. N. Lehmer: *A law of reciprocity for the entries in a table of linear forms.*

In constructing tables of linear forms for use in making his factor stencils the author found the following theorem of prime importance: If A and B are entries in any table, so also is the product AB an entry. This theorem suggested a close analogy between the theory of the entries in these tables and the theory of quadratic residues. This analogy is far reaching. It leads to the following analog of Legendre's Law of Reciprocity: If P and Q are two odd primes, then P is an entry in the table for Q whenever Q in an entry in the table for P unless P and Q are both of the form $4n-1$, in which case if P is in the table for Q, Q will not be in the table for P. Easy modifications are necessary for negative values of P and Q. Linear form tables may also be constructed giving the tables which have a given entry. (Received November 6, 1933.)

2. Professor Morgan Ward: *The diophantine equation* $X^2-DY^2=Z^M$.

All relatively prime integral solutions of the diophantine equation $X^2-DY^2=Z^M$ are obtained by the use of ideals under the assumptions that D is square-free, incongruent to one modulo 8, while M is any positive integer prime to the class number of the quadratic field $K(D^{1/2})$. The formulas correct and extend earlier attempts by Pepin and others to solve the equation, and are applied to discuss the solution of various allied diophantine equations. (Received November 8, 1933.)

3. Dr. D. C. Duncan: *A plane rational curve of order* $2k+1$.

In an earlier paper (to appear in a forthcoming issue of this Bulletin) the writer has established the existence of real, non-degenerate, completely symmetric, self-dual, *elliptic* curves of order $4k+2$, with the singularities all distinct, together with a plan for sketching them approximately. In the present paper one observes that the completely symmetric, self-dual *rational* curve with its singular elements all distinct has a real existence for all orders. Moreover, a very good approximation of such a locus of order n (i.e., $2k+1$) is realized by drawing secant lines through all *consecutive* pairs of $n-2$ points equally distributed around a circle, omitting the chords. The $n-2$ acnodes appear within the circle between consecutive cusps. These rational curves are also invariant under $4k-2$ collineations and $4k-2$ correlations, of which $2k$ are polarities, all of which are listed. The $(2k+1)^2$ foci are shown to admit fairly easy determination. The equations in Cartesian rectangular coordinates are given for the curves of orders 5, 7, and 9, together with a sketch of the locus of order 9, depicting all the singular elements and autopolarizing conics. (Received November 10, 1933.)

4. Professor A. R. Williams: *The apparent contour of the general V_3^n in S_4.*

It is the purpose of this paper to study the surface obtained by intersecting with a 3-space the hypercone of tangents drawn to a hypersurface V_3^n in S_4 from a point O. The two cases when O is not a point of V, and when it is on V, are considered in that order. Results obtained are checked by the known formulas for surfaces in S_3. (Received November 10, 1933.)

5. Professor R. S. Burington: *A classification of plane cubic curves under the affine group by means of arithmetic invariants.*

In this paper a system of arithmetic invariants is exhibited which is sufficient to give a complete separation of real ternary cubics, under the real affine group, into 110 species and 42 canonical forms. Associated with the cubic, C, is a certain set of matrices whose ranks and signatures serve to make the division into species. A prominent part is played by the matrix of the quadratic hessian of C, and by certain quartic covariants and their discriminant matrices. The appearance of certain families of cubics in recent electrical work should add interest to the results of this paper. (Received November 8, 1933.)

6. Mr. P. M. Pepper: *Affine-polygenic functions.*

That the real and imaginary parts of $w(z)=u(x, y)+iv(x, y)$ satisfy at every point in a given region R a pair of partial differential equations of the type $u_x+a_2v_x-a_1v_y=0$, $-v_x+a_2u_x-a_1u_y=0$, a_1, a_2 real, $a_1\neq 0$, characterizes all functions $w(z)$ analytic throughout R in some variable $Z=X+iY$, $X=a_{11}x+a_{12}y+a_{13}$, $Y=a_{21}x+a_{22}y+a_{23}$, a_{ij} real, $a_{11}a_{22}-a_{21}a_{12}\neq 0$. The particular transformation $X=a_1x+a_2y$, $Y=y$ will always yield analyticity. Given $w(z)$, substitution of the specific functions u_x, u_y, v_x, v_y in the above differential equations and solution for a_1 and a_2 in terms of these partial derivatives affords a test for transformability and if $w(z)$ is thus transformable, yields, simultaneously, the constants a_1 and a_2 of a transformation. Integrals of such functions about closed contours are expressible as line integrals and as area integrals of analytic functions: $\int w(z)dz=\pm\frac{1}{2}K\int f(Z)(dX-idY)=\mp iK\iint f'(Z)dXdY$, $f(Z)$ and $f'(Z)$ being, respectively, the transform of $w(z)$ and the derivative of this transform with respect to Z. The first two integrals are taken about, respectively, the original contour and the transformed contour, the third integral is taken over the area enclosed by the transformed contour. The constant K has the value $(1-a_1-ia_2)/a_1$. The upper or lower signs are taken according as a_1 is positive or negative. (Received November 8, 1933).

7. Professor Louis Brand: *A significant vector in the theory of surfaces.*

If a portion of a surface is simply covered by a field of curves of unit tangent vector a, we may form at every point a unique trihedral anb, where n is the unit surface normal and $b=a\times n$. As the point P moves with unit speed along a surface curve of unit tangent t, the angular velocity Ω of the trihedral anb is a function of the position of P and of the direction t. In fact $\Omega=t\cdot\Phi$, where Φ is a dyadic function of position. This dyadic may be expressed as $\Phi=Qn-\nabla n\times n$, where ∇n is the surface gradient of n. This paper deals with the properties of the vector Q thus defined. The integrability conditions for the surface are summarized in $n\cdot\mathrm{rot}\;\Phi=n\cdot\Phi_2$, where Φ_2 is the second of Φ in the sense of Gibbs. In particular $n\cdot\mathrm{rot}\nabla n=0$ gives the Codazzi equations, while $n\cdot\mathrm{rot}\;Q=-K$ gives the equation of Gauss, and at the same time is the vector expression of Gauss' *Theorema Egregium* on the total curvature K. The integral form of this equation, namely $\int K\,d\sigma=-\oint t\cdot Q\,ds$ is precisely *Bonnet's Integral Formula*. (Received November 7, 1933.)

8. Professor C. G. Latimer: *On the units in a cyclic field.*

In this paper three theorems are proved on the existence of certain types of systems of fundamental units in a cyclic field F. If F is imaginary, it is shown that there is a fundamental system in which every unit is real. If F is obtained by composition of two fields F_1 and F_2, it is shown that a fundamental system in F_1 together with a fundamental system in F_2 and certain other units form a fundamental system in F. In another paper the writer expressed the class number of F in terms of an ideal in a certain commutative ring. It is shown that F contains a unit such that certain of its conjugates form a fundamental system if and only if this ideal is a principal ideal. (Received November 10, 1933.)

9. Mr. H. E. Vaughan: *Some theorems on connected sets.*

This paper contains theorems on connected sets and sets irreducibly connected about subsets. Some of these are generalizations of theorems in sections one and two of the paper *Sur les ensembles connexes* by Knaster and Kuratowski (Fundamenta Mathematicae, vol. 2). (Received November 9, 1933.)

10. Professor W. L. Ayres: *On the relation between local connectivity and another property.*

Kuratowski and Hahn have shown that a continuum is locally connected at every point if and only if for each point p every neighborhood of p contains a connected neighborhood of p. In this note these properties are considered for a single point p of the continuum and it is shown they are not equivalent. This question arose in connection with a conjecture of W. A. Wilson (American Journal of Mathematics, vol. 54, p. 382), which is shown to be false. (Received November 9, 1933.)

11. Professor H. S. Pollard: *On the relative stability of the median and arithmetic mean.*

In this paper the relative stability of the median and arithmetic mean of frequency distributions which may be dissected into two and three normal distributions is investigated. The measure of stability employed is the standard deviation of these averages. It is shown that the formula which is ordinarily used to calculate the standard deviation of the median yields only an approximation to its actual value, and that for samples which do not contain a fairly large number of items and for distributions in which the median is located at a point of relatively small frequency this approximation may be untrustworthy. A second method of determining the relative stability of the two averages is then developed. The frequency functions according to which the arithmetic means and medians of samples drawn from a given distribution are distributed are obtained, and the deviations of corresponding percentiles of the two averages are compared. Both methods of determining the relative stability of the median and mean are applied to a particular sequence of economic data. (Received November 27, 1933.)

12. Dr. R. H. Cameron (National Research Fellow): *Algebraic functions of Stepanoff almost periodic.*

In this paper sufficient conditions are given under which a polynomial equation with Stepanoff almost periodic coefficients will have Stepanoff almost periodic solutions. (Received November 8, 1933.)

13. Professor Lincoln La Paz: *On a lemma of Fejér.*

L. Fejér (Mathematische Annalen, vol. 61 (1905), p. 432) has verified and used the following lemma: If $y''=F(x, y, y')$ is the Euler equation of the variation problem $\int_{x_1}^{x_2} f(x, y)\phi(y')dx$, then $y''=-F(x, y, y')$ is the Euler equation of the problem $\int_{x_1}^{x_2} [\phi(y')/f(x, y)]dx$. In the present paper the following generalization of Fejér's lemma is obtained: If for a non-singular problem of minimizing an integral $\int_{x_1}^{x_2} f(x, y_1, \cdots, y_n)\phi(y_1', \cdots, y_n')dx$ with integrand function of class C'' the Euler system in normal form is $y_k''=F_k(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$, $k=1, 2, \cdots, n$; then for a problem of minimizing the integral $\int_{x_1}^{x_2} \Phi(f)\phi(y_1', \cdots, y_n')\, dx$, where Φ is of class C'' but is otherwise an arbitrary function of its argument, the Euler system in normal form is $y_k''=(f\cdot\Phi'/\Phi)F_k$, $k=1, 2, \cdots, n$. The generalized lemma is of service in writing down by inspection the normal Euler equations of the important class of brachistochronic integrals $T=\int_{s_1}^{s_2} [ds/v(r)]$ by reference to the corresponding equations derived in the simple special case $v(r)=r^{-1}$. A generalization of Fejér's lemma for double integrals of the type $\int\int f(x, y, z)\phi(p, q)dxdy$ is also indicated. (Received November 8, 1933.)

14. Professor G. Y. Rainich: *Singularities on maps of surfaces.*

A map is obtained by introducing on a surface and on a plane curvilinear coordinates and representing points of the surface by points of the plane with the same coordinates; the coefficients of the two fundamental forms, given as functions of coordinates, determine the surface. Do closed curves on the map surrounding a singular point of the coefficients correspond to closed curves on the surface? This question is settled by product integrals. Assuming an affirmative answer an invariant is introduced, the number of interlinkings of two surface contours corresponding to two neighboring map contours. First the situation in the case of a ruled surface formed by the normals to a closed curve is studied. Here the invariant is, in the case of a plane contour, an integral of the expression whose vanishing means that the ruled surface is developable. This result is used to express the invariant in the general case as a contour integral involving the fundamental forms; the integrand, in the case of a plane contour, may also be described as the difference of the two principal curvatures times the sine of the double angle between the direction of the contour and a principal direction. (Received November 9, 1933.)

15. Professor I. A. Barnett: *Transformation of an integro-differential equation by means of a Volterra transformation.*

The author shows that certain classes of linear integro-differential equations may be transformed into linear differential equations by means of a Volterra transformation. This leads at once to an existence proof for the solutions of such equations. In fact, an explicit expression is found for the solution of the integro-differential equation in terms of the infinitesimal transformation which generates the finite Volterra transformation. (Received November 10, 1933.)

16. Mr. H. S. Kaltenborn: *Some properties of Stieltjes integrals.*

The definition of Stieltjes integral has been modified and extended in various ways, and different types of Stieltjes integrals have been obtained. This paper is concerned with the four types discussed by H. L. Smith (Transactions of this Society, vol. 27 (1925), p. 491–515), which may be grouped in pairs as "ordinary" and "mean" integrals according to the way in which the approximating sum is formed, the two integrals in each group being distinguished by the method used in passing to the limit. It is shown that the more important elementary properties of the ordinary integrals are also valid for the mean integrals. A fundamental theorem is established for the mean integrals, by means of which a theorem corresponding to the substitution theorem for the ordinary integrals is derived. Several related properties, which are valid for both the ordinary and mean integrals, are then derived from the substitution theorem. The paper concludes with some sufficient conditions for the existence of the mean integrals; a particular result being that the integral of $f(x)$ with respect to $\phi(x)$ exists if $f(x)$ and $\phi(x)$ are any two regular functions of bounded variation. (Received November 10, 1933.)

17. Professor C. N. Moore: *On convergence factors for double series that are summable of non-integral orders.*

In many questions of analysis it is of importance to determine how the introduction of certain factors in the terms of a series affects the behavior of the series as to convergence, summability, or other general properties. Factors that maintain convergence for convergent series or produce convergence in the case of summable series are naturally designated convergence factors. In a previous paper (Transactions of this Society, vol. 29 (1927)) the author has obtained necessary and sufficient conditions for convergence factors in multiple series that are convergent or summable by Cesàro means of integral orders. Interesting and important applications of some of these results to the theory of multiple factorial series may be found in a recent paper by C. R. Adams (Annals of Mathematics, vol. 32 (1931)). The aim of the present paper is to extend the theorems of the 1927 paper that concern double series to the case of series that are summable by means of non-integral orders. It is found that if the double differences occurring in the conditions of the previous theorems are suitably generalized to differences of non-integral orders, the generalized theorems may be stated in essentially the same form. The proofs, however, are considerably more complicated. (Received November 7, 1933.)

18. Miss Alta Odoms: *On the Cesàro mean of double Fourier series.*

This paper is an extension to functions of two variables of some theorems proved by O. Szász concerning the Cesàro mean of the rth order $(0<r\leqq 1)$ of the Fourier development of a function of one variable. Let the function $f(x, y)$ be integrable (L) and finite in a cross neighborhood of the point (x_0, y_0) under consideration, and let the Cesàro mean of the rth order $(0<r\leqq 1)$ of its Fourier development be denoted by $S_{mn}^{(r)}(x, y)/C_{mn}^{(r)}$. Further, let α and β be quantities such that $0<\alpha\leqq r\leqq 1$, $0<\beta\leqq r\leqq 1$. Various results are obtained concerning the limits of the products of $S_{mn}^{(r)}(x, y)/C_{mn}^{(r)}$ by $m^\alpha n^\beta$, $m^r n^\beta/\log m$, $m^\alpha n^r/\log n$, and $m^r n^r/\log m \log n$, respectively, as m and n become infinite. (Received November 7, 1933.)

19. Dr. E. S. Akeley: *An invariantive characterization of quadratic differential forms.*

Consider two quadratic differential forms: $g_{ij}dx^idx^j$ and $h_{ij}dx^idx^j$ $(i, j=1, \cdots, n)$. Let $(\lambda_1, \cdots, \lambda_n)$ be the characteristic values of λ for the pencil $h_{ij}-\lambda g_{ij}$. If $\partial(\lambda_1\cdots\lambda_n)/\partial(x_1\cdots x_n)\neq 0$, then $(\lambda_1, \cdots, \lambda_n)$ will form a canonical coordinate system for this pair of forms. If, furthermore, h_{ij} be derived from g_{ij}, then $(\lambda_1, \cdots, \lambda_n)$ can be considered as a canonical coordinate system for $g_{ij}dx^idx^j$. The case where h_{ij} is the contracted Riemann-Christoffel tensor for g_{ij} was considered in a previous paper (Abstract No. 39-5-147), where the properties of these forms in the neighborhood of the origin were considered. In the present paper, exact solutions corresponding to non-degenerate quadratic forms have been found. (Received November 13, 1933.)

20. Mr. E. H. Hanson: *A new proof of a theorem of Denjoy, Young, and Saks.*

It was shown by A. Denjoy (Journal de Mathématiques, (7), vol. 1 (1915), p. 105) that at every point, with the possible exception of a set of Lebesgue measure zero, the Dini derivatives of a continuous function satisfy one of the following relations: (1) all four derivatives are finite and equal; (2) the two upper derivatives are $+\infty$, and the two lower derivatives are $-\infty$; (3) the upper derivative on one side is $+\infty$, the lower derivative on the other side is $-\infty$, and the other two derivatives are finite and equal. This theorem was shown to hold for measurable functions by G. C. Young (Proceedings of the London Mathematical Society, (2), vol. 15 (1916), p. 360). It was partially extended to unrestricted functions by S. Banach (Comptes Rendus, vol. 173 (1921), p. 457) and was first completely extended by S. Saks (Fundamenta Mathematicae, vol. 5 (1924), p. 98) who based his proof on the corresponding theorem for monotone functions (H. Lebesgue, *Leçons sur l'Intégration* (1904), p. 128). Another proof, as yet unpublished, is due to H. Blumberg who makes use of the result for measurable functions. The present proof, assuming no knowledge of the result for special cases, proceeds along the most natural and direct lines and shows that no simplification is gained by formulating the proof for a restricted class of functions rather than for the general case. (Received November 13, 1933.)

21. Professor O. E. Brown: *The equivalence of n-tuples of matrices.* Preliminary report.

In the non-singular case, the equivalence, in the field of their elements, of two n-tuples of square matrices of order m depends upon the solution of a system of $m^2(n-1)$ linear homogeneous equations in m^2 variables. The variables may be arranged as the elements of a square matrix of order m and a given solution of the system called a solution matrix. The given n-tuples are equivalent if, and only if, the system of equations has a non-singular solution matrix. (Received November 21, 1933.)

22. Dr. W. E. Maier: *Quadratic integral equations.*

F. Bernstein and S. Doetsch in 1922 established an integral addition theorem for elliptic modular functions. Generalizing those results to elliptic theta functions $\sum_{h=-\infty}^{+\infty} e^{2\sigma i h z+\sigma i h^2\omega}=\theta(z, \omega)$ it was found convenient to introduce the *normalized* quantity $t^{-1/2}\theta(z, i/t)=\tau(z, t)$. The latter admits of an identical relation with respect to z and t, viz. for $0<\Re(t)$, $\int_0^1 d\rho\,\tau(z, t-\rho t)[\tau(z, \rho t)-\frac{1}{2}\tau(0, \rho t/4)+\frac{1}{2}]=\tau(z, t)$. (Received November 20, 1933.)

23. Mr. W. H. Ingram: *On the dynamical theory of electrical commutator machines.*

Following Kron, we use tensor notation and write the equations of motion for a slip-ring machine in the form (1): $e_i=R_{ij}\dot{q}^j+L_{ij}(d/dt)\dot{q}^j+\Gamma_{i;mn}\dot{q}^m\dot{q}^n$, where $\Gamma_{i;mn}$ is the ordinary Christoffel symbol and where the coordinates are true. A transformation to quasi-coordinates, (2): $\dot{\xi}_i=\alpha_j^i(\theta)\dot{q}^j$ $(j=1, 2, \cdots, n-1)$, $\dot{q}^j=\beta_k^j(\theta)\dot{\xi}^k$ $(k=1, 2, \cdots, n-1)$, $\xi_n=q^n=\theta$, where θ is the privileged rotor position coordinate, leads to the system (3): $e_\sigma=r_{\alpha\sigma}\dot{\xi}^\alpha+l_{\alpha\sigma}(d/dt)\dot{\xi}^\alpha+\lambda_{\sigma;\beta\gamma}\dot{\xi}^\beta\dot{\xi}^\gamma+l_{\beta s}\beta_{k\sigma}\beta_{n\gamma}\,(\partial\alpha_{ks}/\partial\theta)\dot{\xi}^\beta\dot{\xi}^\gamma$, where $\lambda_{\sigma;\beta\gamma}$ is also a Christoffel symbol of the first kind. The system (3) is known to be identical with that for a commutator machine with an infinite number of commutator segments. It is now found to be identical with the equations of motion of a Lagrangean system whose kinetic energy is expressed in terms of the ξ's and θ and when the ξ's are regarded as moving coordinates. This suggests that the relative angular velocity of commutator and brush is a dynamical homologue of the angular velocity of a system of reference axes and that the last term in (3) may be called the Coriolis voltage. (Received November 20, 1933.)

24. Dr. G. G. Estes: *The lift and the moment of an arbitrary aerofoil Joukovsky potential.*

In this paper is taken up the problem of finding the lift and the moment of an arbitrary aerofoil in a flow which is according to Joukovsky's theory. Given an arbitrary aerofoil, we first select a Joukovsky profile which is near the given profile. We then calculate the Green's function (pole at infinity) for the arbitrary profile by use of the following formula due to Hadamard: $\delta G_\infty^A = -(1/(2\pi))\oint(dG_\infty^M/dN)\ (dG_M^A/dN)\delta N ds$. G_∞^A is the value at the point A of the Green's function with pole at infinity. The path of integration is the Joukovsky aerofoil. δN is the normal variation from the Joukovsky aerofoil to the arbitrary aerofoil. A new labor-saving method of evaluating the above integral is given, and is carried through for an arbitrary aerofoil. We get the value of the new Green's function (G') at the points A by adding the variations given by the integral above to G_∞. Using a Fourier series expansion of G', we get the function conjugate to G' (call it H'). The function that maps the region exterior to the arbitrary aerofoil on the interior of the unit circle is now known, and it is now possible to get the flow about the arbitrary aerofoil. The lift and the moment are then calculated by well known formulas of aerofoil theory. (Received November 29, 1933.)

25. Mr. W. L. Morris: *A new method for the evaluation of double integrals.*

The evaluation of $\iint_A f(x, y)\, dy\, dx$ is not always possible by simple integration. In order to take care of the cases where simple integration is impractical, several methods have been devised. One is that of transformation, where a substitution is made on x and y in an attempt to simplify the limits of integration. This ordinarily renders the integrand more complicated. If, however, we place (1) $f(x, y) = J$ where J is the Jacobian of u, v with respect to x, y, it is possible to find $u = u(x, y)$ and $v = v(x, y)$ satisfying (1). Then $\iint_A f(x, y)\, dy\, dx = \iint_{A'} du\, dv$. This is a simple area integral over A', the evaluation of which may be accomplished readily. One method that suggests itself is the graphical method. Suitable transformations for some of the more common engineering integrands have been obtained. It has been found that the application of some transformations require Riemann surfaces. The evolution of suitable mapping rules for each transformation, however, will permit the use of this method without knowledge of such surfaces. (Received November 25, 1933.)

26. Dr. Wladimir Seidel: *An extension of Cauchy's theorem.*

Let C be a closed rectifiable curve (not necessarily a Jordan curve) which forms the boundary of a simply connected region R in the z-plane. Denote by $w = g(z)$ an analytic function in the region R which maps this region conformally on the circle $|w|<1$. The inverse function $z=h(w)$ of $w=g(z)$ is continuous in the closed circle $|w|\leqq 1$. If the circumference $|w|=1$ is described in the counter clockwise sense, the corresponding point z will describe C in a certain orientation which we call the *simple orientation* of C. Under these conditions the extension of Cauchy's theorem may be stated as follows: *Let $f(z)$ be a function analytic interior to R, and bounded there: $|f(z)|<M$, M being a positive constant. Then $f(z)$ has limiting values along almost all normals to the curve C, defining on C a bounded measurable function $f^*(\zeta)$, where ζ is a point of C, and $\int_C f^*(\zeta)\,d\zeta=0$, the integration being taken along C in simple orientation.* In the case that C is a simple Jordan curve the theorem was proved by V. V. Golubev, Master's Thesis, Moscow, 1916. (Received November 29, 1933.)

27. Dr. R. H. Cameron (National Research Fellow): *Linear almost periodic transformations.*

Almost periodic, mono-basal, and elementary almost periodic transformations are the analogues in transformations of almost periodic, limit periodic, and actually periodic functions. In a paper to appear in the April number of the Transactions of this Society, the author has shown that an almost periodic transformation can be expressed as an infinite product of permutable mono-basal transformations. In the present paper it is shown that a linear almost periodic transformation of a Banach space into itself can be expressed as an infinite product of permutable elementary almost periodic transformations. (Received November 29, 1933.)

28. Mr. Jacques Hammond: *On capacity in potential theory.* Preliminary report.

In this paper it is proved that Kellogg's definition of capacity is equivalent to the transfinite diameter definition, which in turn is known to be equivalent to several others. The corresponding two-dimensional situation is also considered, and a new definition is proposed analogous to Kellogg's three-dimensional definition. The capacities of certain sets bounded by circles are obtained. (Received November 27, 1933.)

29. Dr. Hassler Whitney: *A numerical equivalent of the four-color problem.*

Let $(p_1, q_1), \cdots, (p_m, q_m)$ be pairs of positive integers, with each $q_i>p_i$. We say they form an *admissible set* if there are no i and j such that $p_i<p_j \leqq q_i<q_j$. The following theorem is equivalent to the four-color theorem. *If $(p_1^{(k)}, q_1^{(k)}), \cdots, (p_{m_k}^{(k)}, q_{m_k}^{(k)})$ $(k=1, 2)$ are two admissible sets, and n is the largest of the numbers $p_i^{(k)}, q_i^{(k)}$, then there are numbers $a_1, \cdots, a_n$, each equal to* 1, 2, *or* 3, *such that $a_{p_i^{(k)}}+a_{p_i^{(k)}+1}+\cdots+a_{q_i^{(k)}}\not\equiv 0 \pmod 4$, $(i=1, \cdots, m_k;$ $k=1, 2)$.* (Received November 29, 1933.)

30. Professor R. L. Jeffery: *Derived numbers and approximate derivatives of non-measurable functions.*

The problem of the present paper has been considered for measurable functions by J. C. Burkill and U. S. Haslam-Jones, Proceedings of the London

Mathematical Society, (2), vol. 32, p. 346. In that paper they introduce the concept of λ-derivates. In a later paper, Quarterly Journal of Mathematics, vol. 4 (1933), they use this concept, together with the idea of measurability of one set with respect to another, in a study of derivatives for non-measurable functions. The present paper obtains in a simple way the possible distribution of the values of the derived numbers and approximate derivatives of arbitrary finite functions over arbitrary sets. The discussion in no way features λ-derivatives, or measurability of sets or functions, but is based on derived numbers over sets introduced by Saks, Fundamenta Mathematicae, vol. 5, pp. 98–104, and on outer measure and outer density. (Received November 25, 1933.)

31. Dr. F. W. Perkins: *The Dirichlet problem for domains with multiple boundary points.*

This paper contains a treatment of the Dirichlet problem by a method suggested to the author by Professor O. D. Kellogg shortly before his death. A concept is introduced which is somewhat analogous to the concept of "prime ends" as developed by Carathéodory (*Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1913), pp. 322 ff.). In this way a part of the present theory relating to the Dirichlet problem may be extended, for certain types of domains, so as to render possible (in assigning boundary conditions) a discrimination between various modes of approach to a multiple boundary point. (Received November 29, 1933.)

32. Dr. Mabel Schmeiser: *Some properties of arbitrary functions.*

Let $f(x, y)$ be an arbitrary real function, P a point of the xy plane, I_{Pd} the interval determined by the lower and upper bounds of the values of $f(x, y)$ as P is approached along the direction d, and s a given straight line in the xy plane. It has been proved by Henry Blumberg that at every point P of s except possibly at the points of a denumerable set, I_{P_α} overlaps, laps, or abuts I_{P_β} for two fixed directions α and β on the same side of s (Fundamenta Mathematicae, vol. 16 (1930), p. 77). In this paper the two directions are freed of their fixed positions and the result is proved valid for all pairs of directions on the same side of s. An example shows that for any denumerable set D on a straight line there exists a function for which D is the exceptional set. The extension of this theorem to functions of three variables, substituting a planar direction for one of the linear directions and a given plane for the given line s, gives as the exceptional set an exhaustible plane set. (Received November 29, 1933.)

33. Dr. Saunders MacLane: *Abbreviated proofs in logic calculus.*

Any system of mathematical logic must be based upon certain processes of proof which enable new true expressions to be deduced from known formulas. The simplest such processes are syllogism and inference, while substitution with an equation or with an equivalence, the construction of a normal or half-normal form for any expression, and the constructive proof of an existence theorem are other logical processes in common use in mathematics. In this paper, all these processes and many others are defined precisely, and with complete generality, on the basis of an abstract analysis of the character of symbolical expressions. In virtue of these definitions the conclusion of any process is uniquely determined by its premises. Hence in giving any proof it is possible to omit all intermediate conclusions, if only the processes involved are specified. This method of abbreviating proofs makes possible a considerable condensation and clarification of the usual detailed proofs of formal logic, and also serves as an introduction to the study of plans of proofs; that is, of the principles in a proof which determine the choice of the individual steps of that proof. Such a study opens new possibilities for the application of logic to mathematics. (Received November 25, 1933.)

34. Dr. S. S. Wilks: *The independence of estimates of variance in samples from Latin square lay-outs.*

Let X_{uv} $(u, v=1, 2, \cdots, r)$ be the element in the uth row and vth column of a matrix M of observations from a normal population having variance σ^2. Let $\overline{X}_{u\cdot}$ and $\overline{X}_{\cdot v}$ be the means of the uth row and vth column of M respectively, and $\overline{X}$ the mean of all observations. Furthermore, let $X_{uv_{tu}}$ $(u, t=1, 2, \cdots, r)$ be any classification of the r^2 elements $\{X_{uv}\}$ such that for each value of t no two X's lie in the same row or column of M, and let $r\overline{X}_t=\sum_{u=1}^{r} X_{uv_{tu}}$ Then, if v_r, v_c, v_t, and v_e denote R. A. Fisher's estimates of σ^2 from the sets $\{\overline{X}_{u\cdot}\}$, $\{\overline{X}_{\cdot v}\}$, $\{\overline{X}_t\}$ and $\{X_{uv}-\overline{X}_{u\cdot}-\overline{X}_{\cdot v}-\overline{X}_t+3\overline{X}\}$, respectively, it is shown by means of characteristic functions that these four estimates are distributed independently of each other. Certain generalized forms of Latin square lay-outs are also considered, including the "equalized random blocks" lay-out. (Received November 25, 1933.)

35. Dr. H. L. Dorwart: *Concerning certain reducible polynomials.*

This paper discusses the existence of the reducible polynomials mentioned in Theorem 16 of the paper *Criteria for the irreducibility of polynomials*, by H. L. Dorwart and Oystein Ore (Annals of Mathematics, (2), vol. 34 (1933), pp. 81–94). The problem of finding these polynomials is shown to be equivalent to a problem that occurs in the investigation of rapidly convergent series convenient for the computation of logarithms, which in turn is known to be equivalent to a problem in diophantine analysis. (Received November 28, 1933.)

36. Professor I. M. Sheffer: *Concerning some methods of "best approximation".*

Let $\{\phi_n(x)\}$ be a sequence of functions. A "polynomial" of order n is a linear combination of $\phi_0, \cdots, \phi_n$. Let there be given a method whereby for a given function $f(x)$, to each n there is associated an essentially unique "polynomial" $T_n(x)$ of order n. There may be properties of this method which suggest, for $T_n(x)$, the use of the description "best approximation of all polynomials of order n." Interesting examples are the well known least-square method, and the "best approximation" in the sense of Widder's generalization of Taylor's series. These two examples enjoy the following property: if we inquire into the limit of T_n, and write $T_n = T_0 + (T_1 - T_0) + \cdots + (T_n - T_{n-1})$, then $T_i - T_{i-1} = c_i\psi_i$, where the set $\{\psi_i\}$ is *independent of* $f(x)$, the c_i's being determined by $f(x)$. Then, $T_n = c_0\psi_0 + \cdots + c_n\psi_n$, and when n is increased, terms already present are unaltered. We shall refer to this as the property of *permanence.* This paper considers a large class of methods of "best approximation" (including the two examples mentioned) having the property of permanence. We examine the question of convergence in some cases, in particular lightening some of the conditions of Widder. (Received November 27, 1933.)

37. Professor W. H. McEwen: *Polynomials of best approximation associated with certain problems in two dimensions.*

In this paper two problems are considered, both of which are concerned with the approximate representation of a function $u(x, y)$, by means of polynomials $P_{mn}(x, y)$, throughout a closed region J of the xy-plane bounded by an algebraic curve C. In the first, it is supposed that u vanishes identically on C, and the polynomials P_{mn} are defined so as to vanish identically on C and at the same time minimize the double integral over J of the rth power $(r>0)$ of $|\nabla^2(u-P_{mn})|$. In the second, u is permitted to assume arbitrary values on C, and the polynomials P_{mn} are defined so as to minimize an expression involving not only the integral mentioned above but also the maximum value of $|u-P_{mn}|$ on the curve C. In both cases results are developed concerning the convergence of the polynomials P_{mn} towards the value u as m and n both become infinite. (Received November 25, 1933.)

38. Dr. D. C. Lewis, Jr. (National Research Fellow): *On the periodic motions of dynamical systems with n degrees of freedom.*

This paper is in close relation with a paper by Birkhoff and Lewis (Annali di Matematica, (4), vol. 12), in which was proved the existence of infinitely many periodic motions in the neighborhood of a periodic motion of general stable type. In both papers the critical points of a certain function defined on an n-dimensional torus and a certain symmetric square matrix (c_{ij}) of invariants of the Hamiltonian equations play important roles. In the present paper, a closer study of these yields (under suitable conditions of generality) information concerning the characteristic exponents of the periodic motions, the mere existence of which was established in the Birkhoff-Lewis paper. The results are especially precise in the case when $\sum c_{ij}u_iu_j$ is a definite quadratic form. In this case it is proved among other things that there are infinitely many periodic motions of stable type (that is, with pure imaginary characteristic exponents) near the given periodic motion. (Received November 18, 1933.)

39. Professor C. O. Oakley: *Semi-linear equations.* Part III: *Convex polygons.*

It is found that any convex polygon, treated as a closed curve, can be represented by a semi-linear equation, that is, an equation of the form $u_0 + \sum_{i=1}^{n} m_i|u_i| = 0$, where the m's are constants, and where $u_i = a_ix + b_iy + c_i$, $i = 0, 1, \cdots, n$. For a p-gon $(p>3)$, a value of $n = p-2$ is sufficient (semi-linear representations of polygons need not be unique). However, for the case of the triangle, $p=3$, the least permissible value of n is $n=6$. (See abstracts Nos. 37-5-179 and 38-1-21, for Parts I, II, *Regular polygons* and *Irregular polygons* respectively.) (Received December 1, 1933.)

40. Professor L. W. Cohen: *Lagrange multipliers for functions of infinitely many variables*

The existence of the Lagrange multipliers at a maximum of a function of infinitely many variables subject to an infinite set of auxiliary conditions is established. Lemmas on normal infinite determinants and associated linear systems of equations are proved, one of which incidentally removes a redundancy in an implicit function theorem for functions of infinitely many variables. (Received December 1, 1933.)

41. Professor C. N. Moore: *On criteria for Fourier constants of L-integrable functions of two variables.*

In a note published in the Proceedings of the National Academy of Sciences for September, 1933, the author has given a criterion for Fourier constants of L-integrable functions which is more general than several previously known criteria. In the present paper an analogous criterion for the Fourier cos-cos coefficients of L-integrable functions of two variables is obtained. For a doubly infinite set of constants, a_{mn}, approaching zero when m and n become infinite simultaneously, or when either index becomes infinite while the other remains fixed, one further condition is sufficient that the series $\sum a_{mn} \cos mx \cos ny$ shall converge in the region $(0<x\leqq 2\pi,\ 0<y\leqq 2\pi)$ and define there an L-integrable function of x and y. If we represent by $\Delta_{r+1,r+1}a_{mn}$ the double difference of order $(r+1)$ of the given constants which begins with the term a_{mn}, our further condition is the requirement that the double series whose general term is $m^r n^r |\Delta_{r+1,r+1}a_{mn}|$ shall converge for any $r>0$. The criterion thus obtained is more general than the criteria given by the author in the second volume of the Proceedings of the International Mathematical Congress at Zurich. (Received November 29, 1933.)

42. Professors Salomon Bochner and H. F. Bohnenblust: *Analytic functions with almost periodic coefficients.*

The authors discuss the analytic functions defined by power series of the form $\sum_0^\infty f(n)z^n$, where $f(t)$ is a uniform almost periodic function of t. When $f(t)$ reduces to an exponential polynomial, $f(t)=\sum_1^N a_\rho e^{-i\lambda_\rho t}$, $G(z)$ is the rational function $\sum_1^N a_\rho(1-ze^{-i\lambda_\rho})^{-1}$, whose poles are simple and lie on the unit circle. In the general case the authors obtain formally an infinite series, which is not necessarily convergent. They show, however, that the analytic character of $G(z)$ is precisely the same as if the series were convergent. The method used consists in replacing the multiplier $(1-ze^{-i\lambda_\rho})^{-1}$ by a more suitable function, different from it only in those intervals for λ which contain no exponent of $f(t)$. In the same manner is treated the Dirichlet series $\sum f(\log n)\, n^{-s}$ and the Laplace integral $\int_0^\infty e^{-st}f(t)dt$. The results are similar in all cases; in the last, for example, the following is obtained. The analytic function $G(s)$, the continuation of $\int_0^\infty e^{-st}f(t)dt$, $\sigma>0$, is uniform. Its singular points fill out exactly the closure of the set $\{\lambda_p\}$, where $-\lambda_p$ are the exponents of the u.a.p. function $f(t)$. Isolated singularities are simple poles and correspond to the isolated exponents of $f(t)$. (Received December 1, 1933.)

43. Professor C. R. Adams and Mr. J. A. Clarkson: *Properties of functions $f(x, y)$ of bounded variation*

We study mainly the properties of additivity, continuity, differentiability, measurability, integrability, etc., of functions $f(x, y)$ of bounded variation in the senses of Vitali, Hardy, Arzelà, Pierpont, Fréchet, and Tonelli. Let the classes of functions satisfying the respective definitions be denoted by V, H, A, P, F, and T; our results then include the following. Functions in V, H, A, P, or F are additive; if both f_1 and f_2 are in H, A, or P, $f_1 \cdot f_2$ is in the same class; if f is in H, A, or P and is bounded away from zero, $1/f$ is in the same class; these properties are not enjoyed by the unspecified classes. If f is in P (or A or H), it is continuous almost everywhere and hence R-integrable; V, F, and T contain functions everywhere discontinuous with an arbitrarily large saltus. V, F, and T contain bounded functions that are non-measurable and hence not L-integrable. If f is in P, each of its two *total variation functions* is dominated by a summable function. If only functions continuous in x and in y are admitted, the classes are related in the same way as when only functions continuous in (x, y) are admitted; if only functions upper (or lower) semi-continuous in x and in y are admitted, the classes are related in substantially the same way as when all functions are admitted (see Clarkson and Adams, Transactions of this Society, vol. 35 (1933), pp. 824–854). (Received November 12, 1933.)

44. Professor C. R. Adams and Mr. J. A. Clarkson: *On convergence in variation.*

Let $\{f_n(x)\}$ be a sequence of functions converging on $a \leqq x \leqq b$ to a limit function $f(x)$ of bounded variation; let the total variation of $f_n (n=1, 2, 3, \cdots)$ be denoted by $T(f_n)$ and that of f by $T(f)$; then f_n will be said to *converge in variation* to f on (a, b) when and only when we have $T(f_n)$ on $(a, b) \to T(f)$ on (a, b). The results obtained in this paper include the following. Convergence in variation on (a, b), when f is continuous, implies both $f_n \to f$ uniformly on (a, b) and $T(f_n)$ on $(x', x'') \to T(f)$ on (x', x'') uniformly with respect to x', x'' for $a \leqq x' < x'' \leqq b$; the hypothesis of continuity here imposed upon f can neither be deleted nor weakened to semi-continuity. If $\{f_n\}$ converges in variation to f on (a, b) and f is $>\alpha>0$ (or $<-\alpha<0$), $\{1/f_n\}$ converges in variation to $1/f$ on (a, b). A set of conditions sufficient to insure convergence in variation is given, from which it follows that the partial sum of a power series converges in variation to the sum of the series in any interval interior to the interval of convergence. (Received November 12, 1933.)

45. Professor Philip Franklin: *Derivatives of higher order as single limits.*

The nth derivative of a function at a point may be determined by a single limit involving nth differences or the values at $n+1$ evenly spaced points.

However, this limit may exist without the corresponding derivative. It is here shown that if the analogous expression for unevenly spaced points be used, the existence of the limit is a necessary and sufficient condition for the existence of a continuous nth derivative. Applications to finite Taylor developments are considered. (Received November 28, 1933.)

46. Dr. E. K. Haviland: *On H-functions and the distribution functions associated with them.*

Let $x_1(t), \cdots, x_m(t)$ be real functions bounded in $(-\infty, \infty)$ and measurable in every finite interval. It may be shown that if the curve $X(t)$: $x_1 = x_1(t), \cdots, x_m = x_m(t)$ is an H-function in the sense of Wintner, that is $\lim_{T\to\infty}\int_{-T}^{T}[x_1(t)]^{n_1} \cdots [x_m(t)]^{n_m}dt$ exists for all non-negative integers $n_1, \cdots, n_m$, then there exists a unique function $\phi(E)$ representing the asymptotic relative frequency of the curve in the set E. Conversely, if $X(t)$ possesses such a distribution function $\phi(E)$, $X(t)$ is an H-function, so that the H-condition is a necessary as well as a sufficient condition that the curve possess a distribution function. This, together with the result of Birkhoff's general ergodic theorem, viz., that for dynamical systems possessing an integral invariant the H-condition is necessarily fulfilled for all solutions up to those of a space probability zero, shows that the H-condition is not as special as it appears. If $\phi(E) \neq 0$ for all sets E containing points of the curve in the m-space, $X(t)$ is called a proper H-function. Not all H-functions are proper, but if the $x_i(t)$ are almost-periodic, then $X(t)$ is a proper H-function, and this is true also of the functions considered by Birkhoff. (Received November 25, 1933.)

47. Professor V. C. Poor: *Polygenic functional solutions of certain types of integral equations.*

In this paper the uniqueness and the existence of the solution of integral equations of the Fredholm type are given. The region of integration is a complex domain or area, or a combination of both, and the solution is a polygenic function, in general non-analytic. (Received November 22, 1933.)

48. Dr. W. R. Thompson: *On the tetradite Ψ-function with application to apportionment theory.*

Defining the Ψ-function by $\Psi(r, s, r', s') \equiv [(r+s+1)!(r'+s'+1)!/(r!s!r'!s'!)]\int_0^1 x^r(1-x)^s\int_x^1 y^r(1-y)^{s'}dydx$, we here show that $\binom{r+s+r'+s'+2}{r+s+1} \cdot \Psi(r, s, r', s') \equiv \sum_{\alpha=0}^{r'}\binom{r+r'-\alpha}{r}\binom{s+s'+1+\alpha}{s} \equiv \sum_{\alpha=0}^{\alpha\leqq s, r'}\binom{r+r'+1}{r'-\alpha}\binom{s+s'+1}{s-\alpha}$, and that Ψ is identical with the probability that drawing at random without replacements from $r+s+1$ *white* and $r'+s'+1$ *black* balls we shall draw $r+1$ *white* before $r'+1$ *black*. In an article *On the likelihood that one unknown probability exceed another in view of the evidence of two samples* (Biometrika, in press), the author has introduced a method of apportionment based on the Ψ-function. The ball-draft theorem permits the use of a machine in such apportionment which may be generalized for the case of an arbitrary number of rival *treatments*. It is further shown that $\Psi(r, s, r', s') \equiv \Psi(r, r', s, s') \equiv \Psi(s', r', s, r) \equiv 1 - \Psi(r', s', r, s)$. (Received November 14, 1933.)

49. Dr. W. R. Thompson: *Recursion formulas for the tetradite Ψ and the incomplete I-function.*

The object of this paper is to establish first the relation $\Psi(r, s, r', s') \equiv \Psi(r-1, s, r', s') - (s+s'+1) \binom{r+r'}{r} \binom{s+s'}{s} / \left[(r+s+1) \binom{r+s+r'+s'+2}{r+s+1}\right]$, and secondly the relations $I_p(r+1, s+1) \equiv I_p(r, s+1) - \binom{r+s}{s} p^r (1-p)^{s+1} \equiv I_p(r+1, s) + \binom{r+s}{r} p^{r+1} (1-p)^s$, where $I_p(u, v)$ is the function of Pearson and Müller. The difference formulas for the other variables in Ψ are readily obtained by use of the permutations previously given, and the I-function formulas extended throughout their analytic continuation if the binomial coefficient symbol is generalized to include the meaning of the corresponding Gamma function ratio. (Received November 14, 1933.)

50. Dr. W. T. Reid: *An auxiliary theorem associated with the calculus of variations.*

For the problems of Lagrange, Bolza, and Mayer in the calculus of variations, the following theorem is important: If $E_{12}: y_i = y_i(x)$ $(i=1, \cdots, n;\ x_1 \leqq x \leqq x_2)$ is a normal extremal normal on every sub-interval $x_1x_3 (x_1 < x_3 \leqq x_2)$, and which satisfies the strengthened Clebsch and Mayer conditions, then there exist n mutually conjugate solutions $\eta_{ij}(x)$, $\zeta_{ij}(x)$ of the accessory equations with $|\eta_{ij}| \neq 0$ on x_1x_2. The proof of this theorem originally given by Bliss required E_{12} to have an extension normal on every sub-interval (American Journal of Mathematics, vol. 52, p. 736). Morse gave a proof for the problem of Lagrange that did not require this additional hypothesis (Annals of Mathematics, vol. 32, p. 567). Bliss and Hestenes (Transactions of this Society, vol. 35, p. 319) have used for the Mayer problem essentially the method given by Morse. In the present note this theorem is proved in a very simple manner, depending upon the method originally given by Bliss, together with a suitable modification of the accessory system in a neighborhood of $x = x_1$. The conclusion of the theorem is obtained under the weaker hypothesis that there is an integer $q \geqq 0$ such that on every interval x_1x_3 $(x_1 < x_3 \leqq x_2)$ there are exactly q linearly independent solutions η_i, ζ_i of the accessory system for which $\eta_i \equiv 0$. (Received November 29, 1933.)

51. Professor Wilhelm Maier: *Integral equations and elliptic functions.*

F. Bernstein and G. Doetsch in 1921 found an interesting connection between modular functions and quadratic integral equations. Generalizing those results and representing the elliptic theta functions in normalized form, $t^{-1/2}\sum_h \exp(2\pi izh - \pi h^2 t^{-1}) = \tau(z, t)$ for $R(t) > 0$, we find $\int_0^1 d\rho\tau(z, t-\rho t)\left[\tau(z, \rho t) - \frac{1}{2}\tau(0, \rho t/4) + \frac{1}{2}\right] = \tau(z, t)$. Dealing with analytic functions, defined by integral equations, the role of singular lines appears in a new aspect. Though natural boundaries for analytic continuation, they may admit the existence of certain integral mean values of the function along and even beyond the classical domain of existence. (Received November 28, 1933.)

52. Professor V. C. Poor: *Certain fundamental notions and theorems in polygenic function theory.*

This paper treats two fundamental ideas, regularity and residual values. Sufficient conditions for the validity of the circulation theorems are obtained, and an extension of the circulation theorems to an annular domain. This extension or lemma is used in proving that in the definition for the residue of a polygenic function an arbitrary contour may be used, which in the limit contracts to a point. Finally two theorems are obtained: The residue of a polygenic function, regular in a domain, D, is zero at every point of the domain. Regularity in the Stolz sense is a sufficient condition for an areal derivative in the Pompeiu sense. (Received November 22, 1933.)

53. Professor Dunham Jackson: *Note on relations connecting certain cases of convergence in the mean.*

When a sequence of approximating polynomials or trigonometric sums is determined by a criterion of closest approximation which requires the minimizing of an integral containing a power of the error, theorems on the uniform convergence of the approximation can be regarded in some cases as setting up a relationship between the convergence properties of two measures of the discrepancy between the approximating sum and the function approximated, namely the mean value of the power of the error which enters into the integral on the one hand and the maximum error on the other. This note points out that the proof of such a theorem yields at the same time a comparison of convergence properties in which the two measures of discrepancy involved are the means of two different powers of the error. (Received November 13, 1933.)

54. Professor Marston Morse: *Instability and transitivity.*

A dynamical system is said to be regionally transitive if there exists a motion whose closure is the whole of phase space. The work of Birkhoff, Hopf, Koopman, von Neumann, P. Smith, and others has thrown much light on questions of transitivity. However, it is not yet known in the analytic case whether regional transitivity implies metric transitivity. The previous examples of regional or metric transitivity have been special in character. The present paper shows that a general type of geodesic motion is regionally transitive. The geodesics considered lie on any closed Riemannian 2-dimensional manifold R of genus $p>1$. *A sufficient condition that such a system be regionally transitive is that the geodesics on R be uniformly unstable.* Instability is defined in terms of the equation of normal variation from a given geodesic. In the case where the geodesics on R are uniformly unstable the geodesics whose limit motions include all motions have the power of the continuum. The present paper defines *relative transitivity* and shows that it always exists among the minimizing geodesics of a suitable class Ω on R. When the system is uniformly unstable, Ω becomes the set of all geodesics on R, and relative transitivity becomes regional transitivity. (Received November 29, 1933.)

55. Professor G. C. Evans: *A necessary and sufficient condition for a regular point.*

Let $f(e)$ be a distribution of positive mass, finite in total amount, on a domain T, of boundary t, whose Newtonian potential is $U(M)$. The masses obtained by the sweeping out process, applied to T, according to de la Vallée Poussin converge in the weak sense to a mass $\mu(e)$, and the potential converges to a limiting function $V_0(M)$. Let the potential of $\mu(e)$ be $V(M)$. Then $V(M) \leqq V_0(M)$, but the average of $V(M)$ on any spherical surface is the same as that of $V_0(M)$. Hence it may be deduced that a necessary and sufficient condition that a point Q of t be regular for T is that $V(Q)=V_0(Q)=U(Q)$ for arbitrary $f(e)$. A sufficient condition is a similar equation, where $f(e)$ is a single point mass. (Received December 1, 1933.)

56. Professors Einar Hille and J. D. Tamarkin: *On the theory of Laplace integrals.* II.

Starting from the theory of multiplication of Laplace integrals the authors derive a simple method of analytic continuation of such functions based upon representations of the form $[\mu(z)]^{-1}\int_0^\infty e^{-zu}\, dA_\mu(u)$. Here $\mu(z)\neq 0$ is an absolutely convergent Laplace integral. The choice $\mu(z)=(z+\beta)^{-\alpha}$, β large, gives a method essentially equivalent to the typical means of order α of the first kind. M. Riesz has shown that the corresponding abscissa of convergence $\sigma_0(\alpha)$ is a convex function of α. The same is true for the abscissas of uniform and of absolute convergence, $\sigma_u(\alpha)$ and $\sigma_a(\alpha)$. For the Riemann zeta function $\sigma_a(\alpha)=1-\alpha$, $\alpha\geqq 0$, and $\sigma_u(\alpha)=\frac{1}{2}-\alpha$, $\alpha\geqq\frac{1}{2}$. The interval $0<\alpha<\frac{1}{2}$ offers all the customary difficulties encountered in the critical strip. (Received November 22, 1933.)

57. Professor D. V. Widder: *Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral.*

We prove in this note that the integral equation $\mu(x)=\int_{-\infty}^\infty e^{-xu}d\beta(u)$ has an increasing solution $\beta(u)$ if and only if the double integral $\int\int\mu(x+y)\, g(x)\, g(y)\, dx\, dy$ is positive (definite or semidefinite). (Received November 15, 1933.)

58. Mr. W. C. Randels: *On the summability of the conjugate series.*

In this note a method of summability is exhibited which is regular and $\tilde{F}$-effective without being $\tilde{L}$-effective. The method is similar to that used by Paley, Rosskopf, and the author in the analogous problem for Fourier series, in a paper shortly to appear in this Bulletin. (Received November 22, 1933.)

59. Mr. M. F. Rosskopf: *Some inequalities for non-uniformly bounded ortho-normal polynomials.* Preliminary report.

An important theorem of Young-Hausdorff-F. Riesz on the Fourier coefficients of a function relative to a set of uniformly bounded ortho-normal polynomials fails to hold in the case of certain types of polynomials which are not uniformly bounded. The present paper deals with the extension of this theorem to the cases of non-uniformly bounded ortho-normal polynomials such as Legendre, Jacobi, Hermite, Laguerre, and general Tschebycheff polynomials. (Received November 21, 1933.)

60. Professor A. E. Landry: *On the ordered quadrilaterals in- and circumscribed to the plane rational quartic curve* (R_2^4) *with triangular symmetry*. Preliminary report.

There are two kinds of (ordered) quadrilaterals in- and circumscribed to R_2^4 with triangular symmetry, the one self-symmetric as to a line of symmetry of the curve, the other having no such symmetry. The number, reality, and location of the first kind have been determined by a student of the author and the results published in a doctor's dissertation (June, 1932); the present investigation deals with the same questions for the unsymmetric quadrilaterals. It was found possible to reduce the problem to a final equation in t of the fifth degree. (Received November 27, 1933.)

61. Dr. G. A. Hedlund (National Research Fellow): *Transitive geodesics on closed orientable surfaces of genus greater than one.*

The existence of transitive geodesics on certain closed orientable surfaces of genus greater than one and everywhere non-positive curvature is known. To study the geodesics on such a surface it is desirable to map the covering surface conformally into the unit circle. The existence of transitive geodesics can be proved under the hypothesis that the geodesics and the arcs of circles orthogonal to the unit circle are in one-to-one correspondence in a certain sense. Everywhere non-positive curvature implies this condition, but the converse is not true. The existence of transitive geodesics can then be proved in the presence of regions of positive curvature provided these regions are properly restricted. (Received November 28, 1933.)

62. Professor Philip Franklin: *Geodesics on polyhedral surfaces.*

This paper is concerned with the existence of geodesic lines on a surface of genus zero, having a finite number of conical points. (Received November 28, 1933.)

63. Professor J. L. Synge: *On the deviation of geodesics and null-geodesics particularly in relation to the properties of spaces of constant curvature and indefinite line-element.*

The equation of deviation of geodesics and null-geodesics is obtained very simply. It is applied to give a geometrical interpretation of the Riemannian curvature of a 2-element. For spaces of constant curvature the equation of deviation can be integrated. If the line-element is indefinite, there emanate from a point both convergent and divergent pencils of geodesics, there being convergence only for those directions which make ϵK positive, ϵ being the indicator (± 1) of the pencil (that is, the factor required to make the corresponding value of the fundamental form positive) and K being the constant curvature. It follows that in space-time with signature $+++-$, of constant positive curvature, all space-like geodesics are closed curves of constant length, but time-like geodesics diverge exponentially. Null-geodesics diverge linearly in a space of constant curvature. (Received November 25, 1933.)

64. Dr. W. W. Flexner: *The intersection of chains on a topological manifold.*

In the present paper the author extends his recent investigations (Annals of Mathematics, (2), vol. 32 (1931)) of topological manifolds to the general case, viz., to intersecting chains C_p and C_q $(s=p+q-n>0)$. It is in two parts: (1) the construction on the topological manifold, M_n, of an intersection cycle, Γ_s, using any given covering of M_n by n-cells; (2) the proof that any other cycle, Γ_s', constructed from the same or a different covering is homologous to Γ_s about the geometric intersection of C_p and C_q provided the deformations giving Γ_s and Γ_s' were small enough. (1) Γ_s is constructed piecemeal, using the classical theory on one n-cell of the covering after another. The boundaries of the pieces can be connected by arbitrarily small singular s-chains on M_n giving the cycle. (2) The same sort of construction using chains of a dimension higher gives a chain $C_{s+1}\rightarrow\Gamma_s-\Gamma_s'$. (Received November 8, 1933.)

65. Dr. Deane Montgomery (National Research Fellow): *Operations on plane sets.*

Sierpinski has recently (Mathematica, vol. 5 (1931)) summarized the known results concerning a problem which he formulates as follows: Given a property p of linear point sets and a plane set E, $\Gamma_p(E)$ is the set of all points q on the x-axis such that the vertical line through q cuts E in a set having property p. The problem is to determine the class of $\Gamma_p(E)$ for various classes of sets E and various properties p. The present paper studies as properties p the property of being of the second category, the property of having positive measure, and the property of being at the same time an F_σ and a G_δ. Several different classes of sets E are considered, and an application is made to the theory of implicit functions. (Received November 20, 1933.)

66. Mr. N. E. Steenrod: *Characterizations of certain finite curve-sums.*

The notion of derived aggregates of a closed subset of a compact metric space with respect to a class of closed point sets has been introduced by Whyburn (American Journal of Mathematics, vol. 54, pp. 169–175). He establishes the following: In order that the continuous curve M be expressible as the sum of a finite number of K-curves (where K is some class of continuous curves) it is necessary that, for some integer n, the nth derived aggregate of M with respect to K-curves be vacuous. In the present paper the condition is shown to be sufficient in case K is the class of perfect continuous curves, regular curves, acyclic curves, or boundaries of plane domains. From this it follows that every continuum containing no continuum of condensation and every baum im kleinen curve may be expressed as the sum of two acyclic curves. (Received November 28, 1933.)

67. Professor S. S. Cairns: *Completely regular approximations to regular manifolds.*

For the terms *regular* and *completely regular manifold* we refer to abstract No. 39-5-151, this Bulletin. In the present paper we prove the following approximation theorem. Let M_r be a regular r-manifold in n-space. Then, for any positive values (d, e) however small, there exists a completely regular r-manifold M_r', homeomorphic with M_r in such a way that (1) the distance between any two corresponding points (P, Q) on M_r and M_r' is less than e, and (2) if P is at distance d or more from the irregularities on M_r, then the direction cosines of the tangent r-plane (see *The direction cosines of a p-space in euclidean n-space*, American Mathematical Monthly, vol. 39 (1932), pp. 518–523) to M_r at P differ by less than e from the corresponding direction cosines at Q. The manifold M_r' can be made of class C^m, for any m. With the aid of a polyhedral approximation theorem (abstract No. 39-1-44), the proof is reduced to the case where M_r is a simplicial polyhedron. In that case, M_r' can be made to coincide with M_r at all points of any k-simplex, s_k, which are at distance greater than d from the boundary of $s_k (k=1, \cdots, r)$. (Received December 1, 1933.)

68. Dr. A. H. Black: *Further non-involutorial Cremona space transformations contained in a special linear complex.*

Consider two pencils of surfaces $|F|$ and $|F'|$ of orders n and n', which contain a straight line d as an $(n-1)$- and $(n'-1)$-fold basis element. Make the surfaces of each pencil projective with the points of d. A point P will determine a surface F passing through P, hence a point O on d, and a surface F'. The line PO intersects F' in just one point other than O, the point P' which is the image of P. The general transformation and several special cases are found and discussed. (Received December 1, 1933.)

69. Mr. Garrett Birkhoff: *The topology of transformation-sets.*

The (1, 1) bicontinuous transformations of many spaces, including compact and open manifolds, into themselves, can be connected into continuous groups homeomorphic with subsets of Hilbert space. A study of the explicit content of this fact interrelates closely the convergence properties of abstract spaces and the convergence properties of sets of transformations between them. (Received November 24, 1933.)

70. Mr. Garrett Birkhoff: *Hausdorff groupoids.*

Associated with any Hausdorff space is a topological invariant, namely the Hausdorff space of its $(1\rightarrow 1)$ transformations under a suitable definition of continuity. If the points of the primitive Hausdorff space are isolated, this definition coincides with Kneser's, and we have special applications. Notably, the group of any algebraic extension of an enumerable commutative ring is a Cantor set, metric in the sense of van Dantzig, and possessed of a regular mass-function. Certain related facts concerning groups of importance in algebra are also discussed. (Received November 13, 1933.)

71. Professor Harry Levy: *Linearly connected manifolds and ennuples of curves.*

We give here a treatment of linearly connected manifolds based on ennuples of curves. In place of the coefficients of connection we associate with an ennuple of components ${}_\alpha\lambda^i$ a set of n^3 functions, $\gamma^{\alpha}_{\beta\gamma}$, invariant under transformations of coordinates. If the ennuple is replaced by a new ennuple of components ${}_\alpha\lambda'^i = t_\alpha^\beta\, {}_\beta\lambda^i$, the invariants $\gamma'^{\alpha}_{\beta\gamma}$ are determined by the equations $t^a_\alpha \gamma'^{\alpha}_{\beta\gamma} = \gamma^a_{bc} t^b_\beta t^c_\gamma + {}_b\lambda^j \partial t^a_\beta / \partial x^j$. On the basis of this law of transformation we are able to develop an absolute calculus very similar to that of Ricci. Among the results we obtain is that one can always select an ennuple whose invariants are zero along a curve, whereas a linearly connected space does not necessarily admit a coordinate system in which the coefficients of connection are zero at a point. We set up functions analogous to the curvature tensor and torsion tensor and discover identities which are generalizations of the well known identities $B^i_{jkl} + B^i_{klj} + B^i_{ljk} = 0$ and $B^i_{jklh} + B^i_{jlhk} + B^i_{jhkl} = 0$, where B^i_{jkl} and B^i_{jklh} are the Ricci-Christoffel 4 and 5 index symbols of a Riemannian space. Further results are obtained in the case of pseudo-euclidean space, that is, a space of zero curvature but not necessarily of zero torsion. (Received November 28, 1933.)

72. Professor Harry Levy: *Curvatures in Riemannian space.*

For a curve in ordinary space it is well known that $1/\rho = \lim_{P' \to P} \phi/s$ and $1/\tau = \pm \lim_{P' \to P} \theta/s$, where $1/\rho$ is the curvature, $1/\tau$ the torsion, s the arc between P and P', two points of the curve, ϕ the angle between the tangents at P and P', and θ the angle between the binormals. In this note we give an extension of these relations to a curve in Riemannian space. (Received November 28, 1933.)

73. Dr. L. A. Dye and Professor F. R. Sharpe: *The Bertini transformations of space.*

This paper demonstrates the existence of space involutorial transformations such that in each plane of a pencil of invariant planes there is a Bertini transformation. In one type, 6 of the 8 fundamental points of the Bertini transformation lie on a C_6, $p=3$, the other 2 being either fixed or variable on the axis of the pencil of planes. In another type, 6 of the fundamental points lie on a C_6, $p=4$, and the Bertini transformation is degenerate. A third transformation is discussed in which there is a net of invariant quartic surfaces through a C_{11}, $p=14$. The method used in obtaining this last transformation leads also to an involutorial transformation with a net of invariant surfaces of order $n+1$ through a C_{5n-3} of genus $12n-19$. This type has on each plane of a pencil of invariant planes a Geiser transformation having the 7 fundamental points on the C_{5n-3}. (Received November 10, 1933.)

74. Professor J. L. Synge: *Mechanical models of spaces with positive-definite line-element.*

The object of the present paper is to draw attention to the utility of mechanical models as illustrations of spaces defined topologically and metrically.

The configurations of a dynamical system define the points of the manifold and hence its topology; the kinetic energy defines the metric by $ds^2=2T\ dt^2 =a_{ij}dx^i dx^j$, this being a positive-definite form. A number of examples are given. (Received November 25, 1933.)

75. Professor B. W. Jones: *A test for equivalence of positive quaternary quadratic forms.*

$f=a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+a_{44}x_4^2+2a_{12}x_1x_2+\cdots$ and f' (similarly expressed) are two positive quaternary quadratic forms of the same Hessian. Four new coefficients are defined by $a_{i5}=-a_{i1}-a_{i2}-a_{i3}-a_{i4}$ $(i=1, 2, 3, 4)$. Unitary transformations are given reducing f to a form ϕ, the sum of whose coefficients $-a_{ij}$ $(i\neq j;\ i, j=1, 2, 3, 4, 5)$ is not more than the sum for any form equivalent to f. No more than two coefficients of ϕ can be positive. Similarly find ϕ'. If no coefficient of ϕ is positive, ϕ' is not equivalent to ϕ unless its coefficients are the coefficients of ϕ in some order. If just one (just two) coefficient of ϕ is positive, ϕ' is not equivalent to ϕ unless just one (just two) of its coefficients is positive, all but four of its coefficients are coefficients of ϕ, *and* the sum of its coefficients is the same as the sum of the coefficients of ϕ. If none of the above tests preclude equivalence, the transformations (which are explicitly stated and few in number) which can take ϕ into ϕ' may be tried. (Received November 25, 1933.)

76. Professor E. V. Huntington: *Independent postulates for an "informal Principia system with equality."*

This paper (which is a continuation of a paper presented in June, 1933, abstract No. 39-7-235) adds to the primitive ideas explicitly mentioned in *Principia Mathematica* the idea of "equal in some undefined respect," and discusses as an illustrative example a system $(K, C, +, ', =)$ in which K is the class of "verdicts" pronounced by a court upon a specified group of defendants; C is the subclass of "correct" verdicts; $+$ is "or"; $'$ is "not"; and $=$ means "equal with respect to legal effect." Some correct verdicts are "truistic," and some incorrect verdicts are "absurd," corresponding to the universe element and the zero element of a Boolean algebra, respectively. A set of independent postulates for such a system is given; and the distinction between the "formal" theory and the "informal" theory in the *Principia* is shown to depend upon the inclusion or rejection of a single postulate. Both these papers will appear in full in this Bulletin. (Received November 14, 1933.)

77. Professor Richard Brauer: *Klein's theory of algebraic equations and its connection with the theory of algebras.*

In his *Vorlesungen über das Ikosaeder*, Klein gave a theory of the equations of the fifth degree. Later he treated the equations of the sixth and seventh degrees in similar manner. In his investigations it appears necessary to adjoin certain accessory irrationalities, which are foreign to the problem. Klein was led to them by special geometric devices, which left the nature of these irrationalities not clear. In this paper it is shown that the question can be treated in a natural way with the help of the theory of algebras. Thus one obtains an insight into the nature of the accessory irrationalities. (Received December 1, 1933.)

78. Professor C. G. Latimer: *Note on the class number in a rational semi-simple algebra.*

Let $\mathcal{A}$ be a rational semi-simple algebra of order n, and let $\mathcal{G}$ be a domain of integrity of order n in $\mathcal{A}$, employing Dickson's definition. The finiteness of the number of classes of ideals in $\mathcal{G}$ has been shown (a) by Dr. Shover for left ideals in the case where $\mathcal{A}$ is a division algebra (this Bulletin, vol. 39 (1933), pp. 610–614), and (b) by Artin for right ideals in the case where $\mathcal{G}$ is maximal (Mathematisches Seminar, Hamburg, Abhandlungen, vol. 5 (1927), pp. 261–289). In the present note we prove the finiteness of the left ideal class number without the last mentioned restrictions on $\mathcal{A}$ and $\mathcal{G}$. An application is made to similar matrices. By a result due to Dr. Shover (loc. cit.) the right class number is equal to the left class number. (Received December 1, 1933.)

79. Dr. H. W. Raudenbush: *Ideal theory and partial differential equations.*

The object of this paper is to establish an abstract theory of ideals of partial differential forms that will specialize to results obtained by J. F. Ritt. It is proved that every infinite system of forms contains a finite subset such that every form of the system is a root of a linear combination of forms of the subset with forms for coefficients. There follows a decomposition for ideals of forms that contain all derivatives and all roots that are forms. A solution field is constructed and an abstract Hilbert-Netto theorem is obtained for forms. (Received November 29, 1933.)

80. Dr. D. H. Lehmer: *Lacunary recurrences for Bernoulli numbers.*

Recent work on Fermat's last theorem has made desirable an extension of the tables of Bernoulli numbers. This paper gives a number of explicit recurrence formulas for B_n which are particularly designed to facilitate the calculation of these large numbers. In contrast to the recurrences used heretofore, in which B_n was made to depend on each preceding B, the present recurrences have gaps so that B_n may be computed from those preceding B's whose subscripts are congruent to n with respect to some modulus m. The kth term of the recurrence, then, has the form $B_{n-km}\alpha(n, m, k)c_k$, in which $\alpha(n, m, k)$ is a certain binomial coefficient, and where $c_1, c_2, c_3, \cdots$, is itself a recurring series of fixed order with constant coefficients, and depends only on m. As m is increased the recurrence becomes shorter, while the c's become more complicated. The case of $m=12$ gives the most practical formula. The method of deriving these formulas is quite elementary and makes use of the Blissard-Lucas umbral calculus. Similar recurrences are obtained for Euler's numbers and the coefficients of the tangent function. The method may also be applied to the coefficients of certain elliptic functions. (Received November 29, 1933.)

81. Professor Marie J. Weiss: *On simply transitive primitive groups.*

Certain properties of the transitive constituents of the subgroup that fixes one letter of a simply transitive primitive group are analyzed in this paper. The chief results are embodied in the following theorems: If the subgroup that fixes one letter of a simply transitive primitive group has two transitive constituents of relatively prime degrees m and n, $n>m$, it has a transitive constituent of degree $(>n)$ a divisor of mn. If the subgroup that fixes one letter of a simply transitive primitive group has a regular constituent of degree pq, where p and q are primes, it is of order pq. (Received November 27, 1933.)

82. Mr. M. M. Flood: *Division by non-singular matric polynomials.*

It is shown in this paper how all divisions of a matric polynomial P by a non-singular matric polynomial A may be obtained. Wedderburn has shown the existence of the division transformation in this case, and a simpler proof of this is included here. Some restrictions on the degree and on the coefficients of the quotients are found. The matric polynomial V is said to "reduce" A if VA is of lesser degree than A. The matric polynomial W is said to be "associated" with A if the product WA is of the same degree as A and has leading coefficient unity. All polynomials associated with A are easily found and their degree is unique. Obviously all polynomials which reduce A are determined. (Received November 27, 1933.)

83. Dr. L. M. Blumenthal (National Research Fellow): *A determinant theorem obtained from the characterization of pseudo r-spheric sets.*

The characterization of pseudo d-cyclic sets of points given by the author in two recent papers (American Journal of Mathematics, vol. 54 (1932), pp. 729–738; vol. 55 (1933), pp. 619–640) leads to a geometric proof of the following determinant theorem: *If* $\Delta=|r_{ij}|$, $r_{ij}=r_{ji}$ $(i, j=1, 2, \cdots, n)$, $r_{ii}=1$, *of order* $n>4$, *is such that* (i) *the absolute value of* r_{ij}, $i\neq j$, *is less than* 1, (ii) *every third-order principal minor vanishes,* (iii) *at least one fourth-order principal minor does not vanish, then* (a) *no fourth-order principal minor vanishes,* (b) *each* r_{ij}, $i\neq j$, *has the absolute value* $1/2$, *and* (c) *the value of the determinant is* $-(1/2)\cdot(3/2)^{n-1}(n-3)$. It is suggested that this theorem is the first link of a chain of theorems on determinants of the form Δ of order $n>k+3$, the kth link of which contains the following hypotheses: (i) every principal minor of order less than $k+2$ is positive, (ii) every principal minor of order $k+2$ vanishes, (iii) at least one principal minor of order $k+3$ does not vanish; the conclusions being (a) no principal minor of order $k+3$ vanishes, (b) each element r_{ij}, $i\neq j$, has absolute value $1/(k+1)$, and (c) the value of the determinant is $-[1/(k+1)]\cdot[(k+2)/(k+1)]^{n-1}(n-k-2)$. These theorems do not seem to be in the literature of determinant theory, and no purely algebraic method of deducing them is known to the author. (Received November 8, 1933.)

84. Dr. A. T. Craig: *Correlation of indices.*

If the variables x, y, z, with mean values zero, are normally correlated, it is proved that the indices $u=x/y$, $v=z/y$ are correlated in accord with a Cauchy function with linear regression and with a coefficient of correlation equal to the partial correlation coefficient of x and z. Extensions are made to more than two indices and to the case in which the components of the indices are non-normally correlated but possess linear regression systems. (Received November 25, 1933.)

85. Dr. A. T. Craig: *Note on the moments of a Bernoulli distribution.*

Romanovsky (Biometrika, vol. 15 (1923), pp. 410–412) has given a recursion formula for the moments of the Bernoulli distribution $(q+p)^n$ about its mean np. In the present note, a simpler method of arriving at this formula is given. The method is further used to find a recursion formula for the moments of the Poisson exponential. (Received November 25, 1933.)

86. Dr. Saunders MacLane: *Problems of structure-theoretic type in mathematical logic.*

An analysis of the classical form of mathematical logic shows that this logic is concerned chiefly with the study of the detailed structure of mathematical theorems. This suggests several new types of problems, such as the one of finding standard methods of "translating" ordinary mathematical language into logic. More generally, logic may consider the structure, not only of a single theorem, but also of any body of mathematical doctrine, such as a proof or a theory. In particular, the investigation of the structure of proofs leads on the one hand to an analysis of many "processes" of proof, and on the other hand to the study of purposes of proof. By means of a number of examples, it is shown that such purposes can be objectively analysed into completely general "plans" and "methods." (Received November 25, 1933.)

87. Dr. Saunders MacLane: *General properties of algebraic systems.*

Any algebraic variety, such as a group, a field, a group with operators, and the like, concerns essentially a system composed of a number of functions. If this notion of a system be extended to include relations as well, then we can say that any abstract mathematical theory consists of some system and a number of axioms and theorems about this system. The object of this paper is the investigation of the properties of systems. In the first place, many typical algebraic concepts, such as isomorphism, homeomorphism, direct product, abstraction with respect to a congruence, etc., are really relations between systems and can be defined with complete generality. From this standpoint, we can prove a number of interesting theorems concerning the interconnections of these relations and including as special cases many well known theorems of algebra. One such theorem is the generalization to systems of the "second isomorphism theorem" for groups. Many algebraic theorems may be viewed as special cases of the general theorem that any two isomorphic systems have the same structure. This theorem, as well as a number of similar general theorems, are proved by an application of certain logical concepts; some interesting connections between algebra and mathematical logic are thereby indicated. (Received November 25, 1933.)

88. Professor A. A. Albert: *On a certain algebra of quantum mechanics.*

P. Jordan, J. von Neumann, and E. Wigner have considered certain commutative, non-associative algebras of quantum mechanics. Their algebras have finite order over the field of all real numbers, are real in the sense of Artin, and obey the law $(xy)x^2=x(yx^2)$. In particular, any algebra M of real n-rowed square matrices and with multiplication in M defined as quasi-multiplication $ab=\frac{1}{2}(a\cdot b+b\cdot a)$ of matrices a, b in M satisfies their postulates. Moreover they have proved that every irreducible algebra satisfying their properties is either an algebra M or is the algebra N of all three-rowed Hermitian matrices with elements in the real Cayley algebra of order eight, multiplication being again defined as quasi-multiplication of the matrices. In the present paper I prove that this latter algebra is a new algebra, not equivalent to any M above. Moreover I give a brief proof of the complicated property $(xy)x^2=x(yx^2)$. (Received November 29, 1933.)

89. Professor A. A. Albert: *On cyclic equations of prime degree.*

If $F(x)$ is cyclic of odd prime degree p over F and e is a primitive pth root of unity, then $F(x, e)=F(y, e)$, $y^p=c(e)=c$ in $K=F(e)$, $c\neq d^p$ for any d of K. (Compare Fricke's *Algebra*, I, pp. 413–415.) But a false converse is given in the literature. In fact I prove that necessarily $c(e^t)=d^pc^t$, where d is in K and K is cyclic of degree n over F with generating automorphism $e\longleftrightarrow e^t$. But then conversely if $y^p=c$ as above, the field $F(y)$ is cyclic of degree pn over F and $F(y)=F(e)\times F(x)$, where $F(x)$ is cyclic of degree p over F. I also show how to construct certain quantities c giving all cyclic $F(x)$ so that I have in fact given a formal construction of all cyclic fields of odd prime degree. The results are used to prove that a necessary and sufficient condition that a normal division algebra D of degree p over F be cyclic is that D shall contain a sub-field $F(z)$, $z^p=g$ in F. (Received November 29, 1933.)

90. Dr. R. D. James (National Research Fellow): *The value of the constants in Waring's problem.*

The constants which appear in the Hardy-Littlewood analysis for the number $G(k)$ in Waring's problem were evaluated in a previous paper (abstract No. 39-3-103, this Bulletin). It was then possible to deduce new results for $g(k)$. In another paper (abstract No. 39-1-26, this Bulletin) values for $G(k)$ when k is odd were obtained which were an improvement on those obtained by the Hardy-Littlewood method. It is the purpose of the present paper to evaluate the constants of the second paper and so obtain new results for $g(k)$ when k is odd. It is shown, for example, that $g(7)\leqq 281$, $g(9)\leqq 1023$; that is, *all* numbers are sums of at most 281 seventh powers and of at most 1023 ninth powers. The previous results were $g(7) \leqq 353$ and $g(9) \leqq 1252$. (Received November 29, 1933.)

91. Professor C. H. Forsyth: *An interpolation formula which when adjusted for rational centering of data proves to be osculatory.*

At the Atlantic City meeting of the Society in 1932, the author offered certain modifications for several known interpolation formulas which had the effect of treating given equi-spaced data as areas instead of as values concentrated at the midpoints of the successive intervals. In the present paper it is shown that such modifications for probably the most important interpolation formula (Newton's) prove to be osculatory, that is, each pair of partial interpolation curves for two successive intervals of interpolation have common osculating circles, and therefore the same curvature (including slope) at their point of intersection. Hence, the whole series of interpolations over any number of successive intervals of interpolation will prove relatively smooth from interval to interval. Incidentally, this formula can be used to only third differences and still retain the osculatory property. Heretofore, osculatory interpolation formulas were thought to require fifth differences. (Received November 29, 1933.)

92. Dr. H. L. Krall: *On some asymptotic relations for the characteristic values of elliptic differential equations.*

Let $L(u) = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu$, where $AC - B^2 > 0$ and the coefficients have continuous partial derivatives of the first three orders. Also let $M(u)$ be the adjoint operator. An asymptotic relation is given for the characteristic values of the system $L(u) = \lambda v$, $M(v) = \lambda u$ with the boundary conditions $u = 0$, $v = 0$. A condition is also found for the characteristic values of the system $L(u) = \lambda u$ with $u = 0$ on the boundary. These results are generalizations of the results of Geppert and Gheorghiu who discussed the case $L(u) = u_{xx} + u_{yy} + Du_x + Eu_y + Fu$. (Received December 2, 1933.)

93. Professor R. E. Langer: *The asymptotic solutions of certain linear ordinary differential equations of the second order.*

The ordinary differential equation $du^2/dz^2 + \lambda p_1(z, \lambda)(du/dz) + \lambda^2 p_2(z, \lambda) = 0$ in which λ is a complex parameter and the functions $p_j(z, \lambda)$ are expansible in descending powers of λ includes as special cases many equations of importance. A change of variable gives it the form $d^2u/dz^2 - \{\lambda^2 q_0(z) + \lambda q_1(z) + q_2(z, \lambda)\} u = 0$ in which q_2 is a series in λ^{-1}. If z is real and on an interval in which the determinations of $\{q_0(z)\}^{1/2}$ are distinct, the asymptotic forms of the solutions have been known. This paper considers the equation with the variable z complex and in a region within which $q_0(z)$ has a zero of the second order. The asymptotic forms of the solutions are derived. The paper supplements earlier ones by the author on equations of similar type in which, however, $q_1(z)$ was taken to be identically zero. (Received December 8, 1933.)

94. Mr. E. P. Northrop: *Note on a singular integral.*

This note consists of two theorems regarding convergence in the mean of order 2, as $m\rightarrow\infty$, of the integral $(2\pi)^{-1/2}\int_{-\infty}^{+\infty}f(u)K(x-u;\, m)du$, where $f(u)$ is an arbitrary function belonging to $L_2(-\infty, +\infty)$, and $K(u;\, m)$ belongs to $L_2(-\infty, +\infty)$ for every m. It is found that a set of necessary and sufficient conditions for the convergence of the integral to $f(x)$ [to zero] is as follows: (i) The Fourier transform of $K(u;\, m)$ is less than or equal to a constant M for all m and almost all u. (ii) The Fourier transform of $K(u;\, m)$ converges in the mean of order 2 to 1 [to zero] over every finite interval. (Received December 8, 1933).

95. Professor J. L. Walsh: *Note on the orthogonality of Tchebycheff polynomials on confocal ellipses.*

This note shows that the Tchebycheff polynomials $p_n(z)$, orthogonal on the interval $(-1\leqq z\leqq +1)$ with respect to the norm function $(1-z^2)^{-1/2}$, are orthogonal on every ellipse C whose foci are the points -1 and $+1$ with respect to the norm function $n(z)=|1-z^2|^{-1/2}$, in the sense $\int_C n(z)p_n(z)\overline{p_m(z)}\,|dz|=0$, $m\neq n$. (Received December 15, 1933.)

96. Professor I. A. Barnett and Dr. D. S. Nathan: (National Research Fellow): *Linear transformations in function space.*

The authors establish the group property for the one-parameter family of linear transformations in function-space generated by a given infinitesimal linear integral transformation in this space. They accomplish this without employing computational methods, such as were used by Kennison (National Academy Proceedings, vol. 16, pp. 607–609). They then extend the result to a composite function-space. They also extend to a composite function-space Kowalewski's results on the orthogonal group in function-space (Wiener Sitzungsberichte, vol. 120, II, pp. 101–109). In particular, they obtain the orthogonal group in both finite and infinitesimal form; they show that a given infinitesimal orthogonal transformation generates a one-parameter group of orthogonal transformations, and they obtain the analogue of the Cayley formulas for expressing the coefficients of an orthogonal transformation in terms of skew-symmetric quantities. (Received December 27, 1933.)

97. Dr. Leo Zippin: *Semi-Peanian spaces.*

We investigate in Cauchy spaces (complete metric separable) the notion of semi-compactness: a topologic space is semi-compact if *for every point* x and any U_x (neighborhood of x) there exists a $V_x\subset U_x$ such that the boundary, $F(V_x)$, is compact (or vacuous). We show first that a semi-compact Cauchy space can be compactified by the "addition" of a countable infinity of points. If the space is moreover connected and locally connected we call it semi-Peanian (=quasi-Peanian+semi-compact). These spaces are *characterized* by the following property: a semi-Peanian space C can always be compactified to a Peanian space C^* such that $C^*-C\equiv H$ is a totally disconnected F_σ-set which is *thoroughly avoidable* in C^*. This means that if D is any open connected subset of C^* then $D-D\cdot H$ is connected. It is interesting that C^* is an invariant of C. As consequences of the theorem above, we have an extension to semi-Peanian spaces of most of the theory of Peanian spaces, including the recent work of Claytor on planar Peanian spaces. (Received December 27, 1933.)

98. Dr. E. R. van Kampen: *Topological characterizations of 2-dimensional manifolds.*

Characterizations are given of the 2-sphere, 2-cell with and without boundary, 2-dimensional manifolds without boundary, infinite 2-dimensional manifolds and regions of the sphere. Only those for the two types of general manifolds are new. The others are to be found in two papers by Zippin in the American Journal of Mathematics, vols. 52 and 55, but here the proofs are considerably simplified. The wording of the different characterizations, as well as the proofs, is similar to a very large extent. (Received December 27, 1933.)

99. Professor P. A. Smith: *A theorem on fixed points.*

Let S be a subspace of a cartesian S_m and such that every continuous image in S of every hypersphere of dimension $\leqq pm-m-1$ can be deformed in S to a point; then every topological transformation of S into itself which is of period p leaves fixed at least one point. (Received December 19, 1933.)

100. Dr. Charles Hopkins: *Metabelian groups of order p^m.*

The discussion is restricted chiefly to non-divisible metabelian groups G of order p^m. These are classified primarily according to the number n of independent generating operations and secondarily according to the structure of the commutator subgroup. For $p>2$ it is proved that a basis $s_1, s_2, \cdots, s_n$ always exists such that $s_1^{\lambda_n} s_2^{\lambda_2} \cdots s_n^{\lambda_1}=1$ only when each λ is divisible by the order of the corresponding s. The existence of certain categories of groups G is demonstrated by exhibiting a method of construction. Necessary and sufficient conditions that a correspondence $s_i \backsim \prod_{j=1}^{n} s_j^{\lambda_{ij}}$, $i=1, \cdots, n$, define an automorphism of G are developed. A method for constructing every automorphism of G is given. Several interesting applications of this theory will be presented in a later paper. (Received December 13, 1933.)

101. Dr. G. D. Gore: *On generalized inscribed sequences of surfaces.*

The theory of conjugate nets and of asymptotic families of curves on surfaces was generalized by Bompiani (Rendiconti del Circolo Matematico di Palermo, vol. 46 (1922), p. 91). Generalizations of the classical sequences of Laplace were developed from Bompiani's results by B. Segre (Annales Scientifiques de L'École Normale Supérieure, vol. 44 (1927), p. 152). The present paper generalizes the classical inscribed sequence of Laplace and discovers a large class of sequences more general than those of Segre. Four types of the generalized inscribed sequences are inscribed in the sequences of Segre. The results show that the transformations of Laplace and Segre are applicable to a much less restricted class of surfaces than the class to which they have been applied. (Received December 13, 1933.)

102. Professor W. E. Roth: *On Zehfuss matrices and applications.*

If $A=(a_{ij})$, $(i=1, 2, \cdots, m; j=1, 2, \cdots, n)$, is an $m\times n$ matrix and B is a $p\times q$ matrix, then the $mp\times nq$ matrix $(a_{ij}B)$, whose elements occur in mn blocks of p rows and q columns, each of which is the corresponding element of B multiplied by a_{ij}, is a Zehfuss matrix of A and B (Zehfuss, Zeitschrift für Mathematik und Physik, vol. 3 (1858), p. 298). (Rutherford recently discussed the conditions of equality of such matrices (this Bulletin, vol. 39 (1933), pp. 801–808)). The transformation to certain normal forms and the characteristic values of Zehfuss matrices are studied; also their elementary divisors in certain special cases are taken up when it is assumed that those of $A-\lambda I$ and of $B-\mu I$ are known. Application is then made to the solution of the linear matrix equation $A_1\times B_1+A_2\times B_2+\cdots+A_r\times B_r=C$, and to the solution of the following problem: To write down explicitly the equation $F(x)=0$ of degree mn whose roots are $y_{ij}=\phi(\alpha_i; \beta_j)$ $(i=1, 2, \cdots, m; j=1, 2, \cdots, n)$, where $\phi(\lambda, \mu)$ is a given polynomial in λ and μ and α_i and β_i are the roots of the algebraic equations $f(x)=0$ and $g(x)=0$ respectively. (Received December 13, 1933.)

103. Mr. K. W. Wegner: *Equivalence of pairs of Hermitian matrices.*

Two pairs of Hermitian matrices A, B and C, D are said to be equivalent if and only if there exists a non-singular matrix T such that $\overline{T}'AT=C$ and $\overline{T}'BT=D$. In the case in which $|B|\neq 0$ and $|D|\neq 0$, the necessary and sufficient conditions for the equivalence of A, B and C, D are: (1) the elementary divisors of $A-\lambda B$ and $C-\lambda D$ are the same; (2) the indices of $B(B^{-1}A-\lambda I)^n$ and $D(D^{-1}C-\lambda I)^n$ are the same for all positive integral n and real λ. Condition (1) is not sufficient as has sometimes been stated. The method used is much simpler than that used by Muth (Journal für die reine und angewandte Mathematik, 128 (1905), 302–21) in obtaining the conditions for the real equivalence of pairs of real symmetric matrices. It is based on the most general matrix commutative with the classical canonical form of $B^{-1}A$ and $D^{-1}C$. The conditions are easily generalized to the non-singular case in which neither of the determinants $|\rho A+\sigma B|$ and $|\rho C+\sigma D|$ is zero identically in ρ and σ. The singular case is under consideration. (Received December 8, 1933.)