CONCERNING MAXIMAL SETS*

BY G. T. WHYBURN

1. *Introduction.* Let T be any monotone system of closed subsets of an arbitrary metric space M, that is, any closed subset of a set of the system T is itself a set of T. We shall call a set of the system T a *T-set* or a *set of type T*. We may also think of T as a property such that every closed subset of a set having this property likewise has this property.

Since the null set (0) is a subset of every set, then, whatever be the system T, the null set is always a T-set.

We first establish a general existence theorem for certain maximal sets relative to a system T.

2. THEOREM. *If N is any closed non-degenerate subset of a metric space M such that N is not disconnected by the omission of any T-set, then there exists a maximal subset (a continuum) $H(N)$ of M containing N and having this property.*

PROOF. In the first place, N is connected, since $N-(0)$ is connected and is identical with N.

Secondly, N is not a T-set, for if it were then every closed subset of N would likewise be a T-set, and clearly some closed subset of N disconnects N.

Now if H denotes the sum of all sets N_0 containing N and such that N_0 is not disconnected by the omission of any T-set, then clearly H is connected and hence $\overline{H}$ is a continuum.

We proceed to show that no T-set disconnects $\overline{H}$. Suppose on the contrary, that we have a separation $\overline{H}-T=H_1+H_2$, where T is some T-set in $\overline{H}$. Then $N-N\cdot T$ is connected and thus is contained wholly in one of the sets H_1 and H_2, say H_1. But $H\cdot H_2$ contains at least one point x, since H_2 is open in $\overline{H}$. There exists a continuum N_x in H containing $x+N$ and such that N_x is not disconnected by any T-set. But $N_x\cdot T$ is a T-set and we have a separation

$$N_x - N_x\cdot T = N_x\cdot H_1 + N_x\cdot H_2,$$

and both sets of the right hand member are non-vacuous since

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$H_2 \supset x$ and $H_1 \cdot N_x \supset (N - N \cdot T)$. Thus the supposition that $\overline{H}$ is disconnected by the omission of some T-set leads to a contradiction.

Consequently we have only to set $H(N) = \overline{H}$, and our theorem is established.

3. THEOREM. *The common part of any two sets $H(N)$ is a T-set.*

For let $H(N_1) \neq H(N_2)$ and suppose $H(N_1) \cdot H(N_2) \neq 0$, the other case being trivial. Set $H(N_1) + H(N_2) = H$. There exists a T-set T in H such that we have a separation $H - T = H_1 + H_2$. Now $H(N_1) - T \cdot H(N_1)$ and $H(N_2) - T \cdot H(N_2)$ are connected, since the sets taken away are T-sets. Hence $T \supset H(N_1) \cdot H(N_2)$, for if not then the set

$$[H(N_1) - T \cdot H(N_1)] + [H(N_2) - T \cdot H(N_2)]$$
$$= H(N_1) + H(N_2) - T = H - T$$

is connected. Thus $H(N_1) \cdot H(N_2)$ is a T-set, since it is a subset of T.

4. *Applications.* (i) Let T be the property of being the null set. Then the corresponding sets $H(N)$ given by §2 are the components of M.

(ii) Let T be the property of containing at most n points, $(n = 0, 1, 2, \cdots)$.

For $n = 0$, we have the case just considered under (i). For $n = 1$, we obtain for $H(N)$ subcontinua of M which are maximal with respect to the property of having no cut point.* In case M is a locally connected continuum, these sets $H(N)$ are the true cyclic elements† of M.

For $n > 1$, the sets $H(N)$ are subcontinua of M (hitherto unknown) which are not disconnected by the omission of any n points and are maximal with respect to this property. For example, for $n = 4$, let C_1 and C_2 be concentric circles of radii 1 and 2, respectively; let Q be a square, together with its interior, inscribed in C_1, let I be the annular region between C_1 and C_2, and let $M = C_1 + C_2 + I + Q$. Then M has two sets $H(N)$, namely, $C_1 + C_2 + I$ and Q; and it will be noted that, conforming to §3,

* See my paper, American Journal of Mathematics, vol. 55 (1933), p. 456.

† See American Journal of Mathematics, vol. 50 (1928), pp. 167–194.

their common part contains just four points and hence is a T-set. The situation is exactly the same in this example for any $n>4$.

(iii) Let T be the property of being countable (that is, of power $\leqq \aleph_0$).

As an example, let C_1, C_2, and I be defined as under (ii), let J be any simple closed curve having any countable set of points in common with C_1 and lying otherwise within C_1, let D be the interior of J and let $M=C_1+C_2+I+D$. Then the sets C_1+C_2+I and $J+D$ are the sets $H(N)$ in M.

(iv) Let T be the property of being homeomorphic with some proper subset of a simple closed curve.

As an example for this case, let S denote the surface of the unit sphere in R_3 and let M be $S+$any one-dimensional structure which together with S forms a continuum, for example, the part of the x, y, and z axes lying within S. Then S is the only set $H(N)$ in M.

(v) Let T be the property of being at most $(n-2)$-dimensional, $(n=2, 3, 4, \cdots)$.

In this case the sets $H(N)$ are the so-called "n-dimensional components"* or the "maximal n-dimensional cantorian manifolds"† in M. In this case also it will be noticed that §3 above gives the known fact that the common part of any two n-dimensional components is at most $(n-2)$-dimensional.

(vi) Let T be the property of being the carrier of no essential complete n-dimensional cycle,‡ $(n=0, 1,2, \cdots)$. (We shall consider only non-oriented cycles.)

For $n=0$, we have identically the case $n=1$ considered under (ii), since any set containing more than one point is the carrier of a 0-cycle.§ Thus in case M is a locally connected continuum, we obtain the true cyclic elements of M for the sets $H(N)$.

For $n=1$, let us consider the following example. Let W denote the set consisting of the surface of a torus together with a coaxial disc just fitting into it, (that is, a disc-wheel with tire attached), and let C be the surface of a cube which is attached

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* See Alexandroff, Mathematische Annalen, vol. 106 (1932), p. 215.

† Tumarkin, Comptes Rendus, vol. 186 (1928), p. 420.

‡ See Vietoris, Mathematische Annalen, vol. 97 (1927), p. 458; Alexandroff, Annals of Mathematics, (2), vol. 30 (1928), p. 159.

§ We consider a 0-cycle as any *even number* of points (0-cells).

to W along some simple arc, and let Q be a 2-cell (topological square plus its interior) attached to C along some arc and having nothing in common with W. Then W and C are the sets $H(N)$ for $W+C+Q$.

In general, for any n, examples indicate that in this case the sets $H(N)$ are maximal subcontinua without *edge points*; and in many ways they seem to be true generalizations of the cyclic elements of locally connected continua (to which they reduce in case $n=0$) and hence might well be called the "nth order cyclic elements"* of M. We shall consider these sets further in the next section.

5. *Cyclic Elements of nth Order.* Throughout this section we shall let T be the property considered under (vi) in §4 of being the carrier of no essential complete n-cycle, and we shall suppose M to be compact. We shall show that in this case the sets $H(N)$ in a compact continuum M have the property that if K is any subcontinuum of M such that every n-cycle in K is ~ 0 in K, then every n-cycle in $K \cdot H(N)$ is ~ 0 in $K \cdot H(N)$.

For $n=0$ this gives simply the known fact† that the product of any subcontinuum of M by any set $H(N)$ is either vacuous or connected; and in case M is locally connected, it is a special case of the well known and useful property of cyclic elements that the product of any cyclic element by an arbitrary connected set in M is either vacuous or connected.

Now this result is an immediate consequence of the following somewhat sharper theorem.

THEOREM. *If C is an irreducible carrier of the n-dimensional complete cycle C^n in $H(N)$ and B is any irreducible membrane‡ in M which is a carrier of the homology $C^n \sim 0$, then we have $B \subset H(N)$.*

For $n=0$ this says simply that every irreducible continuum in M between two points of $H(N)$ is contained in $H(N)$, a known result (see my paper, loc. cit.). We shall prove the theorem with the aid of the following lemma.

LEMMA. *If C is any carrier of an n-cycle C^n, if B is an irreducible membrane in M which carries the homology $C^n \sim 0$, and T is*

* See R. L. Wilder, this Bulletin, vol. 38 (1932), p. 678.

† See my papers cited above.

‡ See Alexandroff, loc. cit., p. 179.

any T-set (that is, a set carrying no essential n-cycle) disconnecting B, then for every separation $B-T=B_1+B_2$ we have $B_1\cdot C\neq 0 \neq B_2\cdot C$.

PROOF OF THE LEMMA. Suppose, on the contrary, that for some such separation we have $B_2\cdot C=0$. Then we shall show that B_1+T carries the homology $C^n\sim 0$ so that B is reducible. Now we have $C^n=(z_1, z_2, \cdots)$, and since $C^n\sim 0$ in B, then z_i bounds a δ_i-complex K_i^{n+1} in B and $\lim \delta_i=0$. For each i, let H_i^{n+1} be the complex consisting of all simplexes of K_i^{n+1} which have all their vertices in B_1+T, and let c_i be the boundary of the complementary complex of H_i^{n+1} in K_i^{n+1}. Let us now replace c_i by an n-cycle c_i^* in T as follows. For each vertex x_i of c_i, let us take a point x_i^* in T such that $\rho(x_i, x_i^*)=\rho(x_i, T)$, that is, a point of T which is as near x as any point of T. For each simplex $(x_0, x_1, \cdots, x_n)$ of c_i, let $(x_0^*, x_1^*, \cdots, x_n^*)$ be a simplex. Then the complex c_i^* of all such simplexes is an n-cycle in T. Let ϵ_i^* be the norm of c_i^*, that is, the maximum of the diameters of the simplexes of c_i^*.

Now since each simplex of c_i is on a δ_i-simplex of K_i^{n+1} which has at least one vertex in B_2, it follows that if $d_i=\max \rho(x_i, x_i^*)$, then $\lim d_i=0$. Thus since $\delta_i\rightarrow 0$, it follows that $\epsilon_i^*\rightarrow 0$. Now if δ_i' is the greatest lower bound of the numbers δ such that c_i^* bounds a δ-complex in T, then since T carries no essential n-cycle, it follows by a result of Vietoris† that $\lim \delta_i' =0$.

Let $\delta_i^*=2\delta_i'$. Then for each i, c_i^* bounds a δ_i^*-complex L_i^{n+1} in T and $\lim \delta_i^*=0$. Then‡ $L_i^{n+1}+H_i^{n+1}$ is a $(\delta_i+2d_i+\delta_i^*)$-complex bounded by z_i (mod 2), and this complex is contained in B_1+T. Thus $C^n\sim 0$ in B_1+T, contrary to the fact that B is irreducible; and our lemma is established.

PROOF OF THE THEOREM. We have only to show that $H(N)+B$ is not disconnected by the omission of any T-set. Suppose, on the contrary, that for some T-set T, we have a separation $H(N)+B-T=H_1+H_2$. Then since $H(N)-T\cdot H(N)$ is connected, it must be contained wholly in either H_1 or H_2, say H_1. Thus $C-C\cdot T\subset H_1$ and $H_2\subset B-C$. But we have the separation

† See Vietoris, loc. cit., p. 464.

‡ It is understood that we take here the modified complex H_i^{n+1}, that is, the one obtained after replacing each x_i in c_i by the corresponding x_i^*.

$B - B \cdot T = H_2 + H_1 \cdot B$, and $H_2 \cdot C = 0$, contrary to the lemma.

6. *Conclusion*. In conclusion attention is called to the desirability of clearing up, in the general case, the possibilities for the power of the class of all sets $[H(N)]$ in a compact space for any system T, such as has already been done by Mazurkiewicz and Alexandroff (see papers in Fundamenta Mathematicae, vols. 19 and 20) in the special case of the dimensional components. Also a more detailed study of the structure of continua M of varying degrees of connectivity and local connectivity with respect to the sets $H(N)$, in particular in the case* considered in §5, would be highly desirable.

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INTEGRAL DOMAINS OF RATIONAL GENERALIZED QUATERNION ALGEBRAS†

BY A. A. ALBERT

1. *Introduction*. We shall consider generalized quaternion algebras

$$Q = (1, i, j, ij), \qquad ji = -ij, \qquad i^2 = \alpha, \qquad j^2 = \beta,$$

over the field R of all rational numbers. It is easily shown that, by a trivial transformation on the basis of Q, we may take α and β to be integers without square factors.

Of great interest in the theory of algebras Q are the integral sets of Q. L. E. Dickson‡ has called a set S of quantities of Q an integral set if S satisfies the following postulates:

R: The quantities of S have minimum equations with ordinary whole number coefficients and leading coefficient unity.

C: S is closed under addition, subtraction, and multiplication.

U: S contains 1, i, j.

M: S is maximal.

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* A further study of this case is made in the author's paper *Cyclic elements of higher order*, to appear in the American Journal of Mathematics, vol. 56 (1934).

† Presented to the Society, June 19, 1933.

‡ See Dickson's *Algebren und ihre Zahlentheorie*, pp. 154–197, for his theory as well as references to the work of Latimer and Darkow. See also Latimer's later paper, Transactions of this Society, vol. 32 (1930), pp. 832–846.