THE CATEGORY OF THE CLASS LIP (α, p)

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A function $x(s)$ is said to belong to the class Lip(α, p) on the interval (a, b) provided

$$\|x(s+h)-x(s)\| \equiv \left(\int_a^b |x(s+h)-x(s)|^p ds\right)^{1/p} = O(h^\alpha),$$

where $0<\alpha\leqq 1$.

There exist continuous functions which belong to no class Lip(α, p). Indeed if $x(s)\subset$ Lip(α, p), then the Fourier coefficients of $x(s)$, a_n, b_n, are $O(n^{-\alpha})$. Now a continuous function may be constructed* such that $|a_{n_i}|>1/\log n_i$ for an infinite set of values $\{n_i\}$. Then for such a function

$$\frac{|a_{ni}|}{n_i^{-\alpha}} > \frac{n_i^\alpha}{\log n_i} \neq O(1),$$

that is, $a_n\neq O(n^{-\alpha})$ and hence the continuous function with the Fourier coefficients a_n belongs to no class Lip (α, p).

We prove the following theorem.

THEOREM. *The subset E of L_p, $p\geqq 1$, which is* $\sum$ Lip(α, p) *for* $0<\alpha\leqq 1$, *is of the first category in* L_p.

We employ a method of proof used by S. Banach.† We take the interval (0, 1) as the fundamental interval and assume the functions to be periodic with the period one. Let E_{nm} be the set of all $x(s)\subset L_p$ such that

$$\int_0^1 |x(s+h)-x(s)|^p ds \leqq n^p |h|^{p/m}, \qquad (n, m = 1, 2, \cdots).$$

The sets E_{nm} are closed. For, let $x_i(s)\to x_0(s)$ in L_p. Set

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* W. Randels, *A remark on Fourier series of continuous functions*, American Mathematical Monthly, vol. 40 (1933), pp. 97–99. See also an article by O. Sźász, to appear soon in the same journal.

† *Über die Baire'sche Kategorie gewisser Funktionenmengen*, Studia Mathematica, vol. 3 (1931), pp. 174–179.

$$y_i(s) = x_i(s+h) - x_i(s),$$
$$y_0(s) = x_0(s+h) - x_0(s),$$

where h is fixed but arbitrary. Then

$$\|y_i - y_0\| \leqq \|x_i(s+h) - x_0(s+h)\| + \|x_i(s) - x_0(s)\|$$
$$= 2\|x_i(s) - x_0(s)\| \to 0.$$

But $\|y_i - y_0\| \to 0$ implies that $\|y_i\| \to \|y_0\|$, that is,

$$\int_0^1 | x_0(s+h) - x_0(s) |^p ds \leqq n^p | h |^{p/m}.$$

Moreover $E \subset \sum_{n,m=1}^{\infty} E_{nm}$. For, if $x_0(s) \subset E$, then for some value α_0, $x_0(s) \subset \text{Lip}(\alpha_0, p)$; that is, there exists a number M such that

$$\int_0^1 | x_0(s+h) - x_0(s) |^p ds \leqq M | h |^{\alpha_0 p}.$$

To complete the proof we have only to show that every set E_{nm} is non-dense. Suppose, if possible, that E_{NM} were not non-dense. Then, since E_{NM} is closed, it contains a sphere K. Let $\omega(s) \subset K \subset E_{NM}$ be the center of the sphere and $r>0$ the radius. Let $g(s) \subset L_p$ be a function of E. Since when $g(s) \subset E$, $c \cdot g(s) \subset E$, where c is a constant not zero, we may assume $\|g\| < r$. Also

$$\|g(t+h) - g(t)\| > 2N | h |^{1/M}.$$

Set $z(s) = \omega(s) + g(s)$. Then $z(s) \subset L_p$ and

$$\|z(t+h) - z(t)\| \geqq \|g(t+h) - g(t)\| - \|\omega(t+h) - \omega(t)\|$$
$$> 2N | h |^{1/M} - N | h |^{1/M} \geqq N | h |^{1/M};$$

that is, $z(s)$ not $\subset E_{NM}$. But $\|z - \omega\| = \|g\| < r$; this means $z(s) \subset K \subset E_{NM}$, a contradiction.

In exactly the same manner we may prove the following result.

The subset EC of the space C of continuous functions is of the first category in C.

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