WIENER ON THE FOURIER INTEGRAL

The Fourier Integral and Certain of its Applications. By Norbert Wiener. Cambridge University Press, 1933. xi+201 pp.

An analyst of the older generation would probably understand the term Fourier integral to refer to the formula

$$f(x) = \frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t)\cos u(t-x)dt,$$

which is valid for a very restricted class of functions. He would not find this formula in Wiener's treatise, which is really devoted to the theory of Fourier transforms, and the corresponding reciprocity relations. The Fourier transform of $f(x)$ is

$$g(x) = (2\pi)^{-1/2}\int_{-\infty}^\infty f(t)e^{-ixt}dt \tag{1}$$

with suitable interpretation of the integral which ordinarily does not exist in the sense of Lebesgue. The reciprocity theorem states that conversely $f(x)$ is the Fourier transform of $g(-x)$ with, conceivably, a different interpretation of the integral.

Wiener is concerned almost exclusively with three classes of measurable functions $f(x)$, namely, L_1, L_2, and S. Here the first two symbols have their customary meaning, S is the class of functions of uniformly bounded mean square modulus. In the first case (1) is an ordinary Lebesgue integral, $g(x)$ is continuous and vanishes at infinity. In the second case the integrals are limits in the mean of order two, and the transform belongs to L_2 [Plancherel's theorem]. The transformation is unitary so that $f(x)$ and $g(x)$ have the same norm in L_2. In the third case, the integral of the formal transform serves the same needs.

The book is grouped into an Introduction and four chapters. Of these Chapter 1 uses ideas from a paper by the author on Plancherel's theorem (Journal of Mathematics and Physics, M.I.T., vol. 7 (1928)). Chapters 2 and 3 are an elaboration of portions of the author's Bôcher Prize memoir (Annals of Mathematics, (2), vol. 33 (1932)). The last chapter is based on another paper by the author (Acta Mathematica, vol. 55 (1930)). The Introduction [45 pp.] gives a rapid outline of the main theorems on Lebesgue and Riemann-Stieltjes integrals, orthogonal series, and the Riesz-Fischer theorem. We note in passing that Theorem X_{21} is false, but as the author neither proves nor uses it, no harm is done.

Chapter 1 [26 pp.] is devoted to Plancherel's theorem mentioned above. The proof given by the author (one of his own; he has given several others) is ultimately based upon the fact that the function $\exp(-x^2/2)$ is its own Fourier transform and on related properties of Hermitian polynomials. If $f(x)\epsilon L_2$, and

$$f(x) \sim \sum_{n=0}^\infty f_n\psi_n(x), \tag{2}$$

where $\{\psi_n(x)\}$ is the orthonormal set of Hermitian functions, then

$$g(x) \sim \sum_{n=0}^{\infty} (-i)^n f_n \psi_n(x), \tag{3}$$

the series being Abel summable as well as convergent in the mean. Formulas (2) and (3) show the unitary character of the Fourier transformation at a glance.

Chapter 2 [32 pp.] deals with the *General Tauberian theorem.* This fundamental discovery, which the author made in 1926–27, is concerned with the limiting properties of linear transformations of the type

$$K[f] = \int_{-\infty}^{\infty} K(x-t)f(t)dt, \tag{4}$$

where $K[1]=1$. Suppose that for a bounded function $f(x)$

$$\lim_{x\to\infty} K[f] = A. \tag{5}$$

Can any conclusion be drawn about the existence of such a limit if the kernel $K(u)$ is replaced by another kernel $K_1(u)$? Wiener's answer is that (5) will hold with K replaced by K_1 provided both kernels belong to L_1, and the Fourier transform of $K(u)$ vanishes for no real value of x. Moreover, if the Fourier transform has a real zero, there exists at least one kernel $K_1 \epsilon L_1$, $K_1[1]=1$, for which (5) is not true. The theorem appears to lie quite deep; the proof requires a very elaborate analysis, and led the author to a number of interesting theorems on absolutely convergent Fourier series and closure in the space L_1.

Wiener's basic theorem seems rather far removed from Tauber's original theorem of 1897 or its later refinements, but this is actually the first application in Chapter 3, *Special Tauberian theorems* [46 pp.]. This is followed by various important applications to analytical number theory. The author gives two distinct reductions of the prime-number theorem, in the form due to Hadamard and de la Vallée Poussin, to Tauberian form. This includes a proof of the Ikehara-Landau theorem. He does not expect much from Tauberian methods, however, when it comes to the more refined theorems on the distribution of the primes, in particular those theorems which involve the Riemann hypothesis.

Chapter 4, *Generalized harmonic analysis* [50 pp.], deals mainly with the functions of class S. This discussion really starts in the last paragraph of Chapter 3 where the mean square modulus and an integrated Fourier transform are brought into play. The principal role is played by the function $s(u+\epsilon)-s(u-\epsilon)$, which is the ordinary Fourier transform of $2f(x)\sin \epsilon x/x$ in L_2. The author defines the "spectrum" of $f(x)$ to be the function

$$\sigma(u) = \text{const.} + \operatorname*{l.i.m.}_{\epsilon\to 0} \frac{1}{2\epsilon(2\pi)^{1/2}} \int_0^u |s(t+\epsilon) - s(t-\epsilon)|^2 dt. \tag{6}$$

This expression determines the energy distribution of the oscillation $v=f(t)$, v being the displacement and t the time. If $f(x)$ be subjected to a transformation of type (4), with suitable restrictions on the kernel, a function of class S will again result whose spectrum is easily computed from that of $f(x)$. The local deformation of the spectrum is proportional to the square of the modulus of the Fourier transform of the kernel. These ideas are finally applied to a derivation of the basic theorems on uniformly almost periodic functions.

This rapid survey will give some notion of the bewildering wealth of new ideas in Professor Wiener's book. The reader who approaches this book for the first time should realize in advance that he is not holding in his hands a compendium of the theory of Fourier transforms, complete up to July 1932, but a report from the workshop of one of the investigators who has perhaps done more than anybody else to establish the Fourier integral firmly as an indispensable tool in modern analysis. The book under review contains the main gems which he has produced with his tools up to the date mentioned. We all know that the production is still going on, and the recent Colloquium lectures of the author and the late R.E.A.C. Paley show no signs of any approaching dullness of the tools nor of the wits behind them. It lies in the nature of the subject that the book is not easy reading, but the Introduction and Chapter 1 are well suited for the needs of a student who wants to learn the elements of the theory.

The lamented Paley used to amuse his friends by stating two criteria for estimating the value of a mathematical contribution. According to the first, one computed the essential theorem density, that is, the number of significant theorems divided by the number of pages. In the second criterion, the essential quotation density, the number of theorems was replaced by the number of references to Paley's work! If one of those who also writes on Fourier integrals should apply these tests, mutatis mutandis, to Wiener's book, the score would come out very high according to the first one, but fairly low according to the second.* The book is singularly autarchic. This is in some sense a weakness; the reader would have gained here and there by a broader outlook. But the reader has to keep in mind the author's stated purpose. He wanted to present his own investigations in book form with the prefatory material necessary to follow the argument, and he has succeeded admirably. It is not a text book, nor a monograph on all the central features of the theory of Fourier integrals. But a person who wants to learn what Fourier integrals are, what they are good for, and who aspires to mastery of the new technique involved in handling them, will do well to sit down and read this book patiently.

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* If the reviewer had got his dues on p. 70, he would have been able to raise the second score from 0 to 1/212.