A NEW PROOF OF THE EQUIVALENCE
OF E. H. MOORE*

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The theorem which is here called the equivalence of E. H. Moore is a statement of a necessary and sufficient condition for the existence of a solution of a system of linear equations.

In Moore's own proof of this equivalence, there is a hypothesis to the effect that if the Hermitian square of a vector α' is zero, all the elements are zero. That is,

$$\alpha' \bar{\alpha}_{,} = 0 \cdot \supset \cdot \alpha' = 0'.$$

In plain language, take a set of complex numbers, or numbers in any number system, multiply each by its conjugate, and add; then if the sum is zero, all the original numbers are zero.

This hypothesis is satisfied in many number systems, and holds in every case in which Moore used his equivalence. Unfortunately, the writer† used the equivalence in the number system of residues modulo 2. For such a number system the hypothesis does not hold, as for example in the case of the vector $\alpha' = (1, 1)$, where $\alpha' \bar{\alpha}_{,} = 1 + 1 \equiv 0$ (modulo 2). To form a logical basis for this work, it was necessary to have a proof, such as the following, of the equivalence which would not use the limiting hypothesis.

A general system of linear equations is written

$$\alpha_{,}^{,'} \xi_{,,} = \eta_{,}$$

in matrix form. The prime range (′) and the second range (″) are conceptually distinct, so that the matrix $\alpha_{,}^{,'}$ may be square or not. Let n be the number of rows in the matrix, or the number of equations in the system, and m the number of columns, or unknowns.

The symbol ∃ is read "there exists," and ∋ is read "such that." The statement that a solution of the system of equations exists is written

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$$\exists\,\xi_{,,} \ni \alpha_{,}''\xi_{,,} = \eta_{,}. \tag{X}$$

Two vectors α' and $\beta_{,}$ are said to be *orthogonal* if

$$\alpha'\beta_{,} = 0.$$

A vector α' is orthogonal to all the columns of a matrix $\beta_{,}''$ if

$$\alpha'\beta_{,}'' = 0''.$$

It may be shown by straightforward substitution that

$$X\cdot \supset \cdot Y,$$

where Y is the proposition

$$\gamma'\alpha_{,}'' = 0''\cdot \supset \cdot\gamma'\eta_{,} = 0.$$

That X implies Y and, conversely, Y implies X, constitutes the equivalence of E. H. Moore.* In Moore's proof of the converse, he normalizes the columns of the matrix, and for this step he needs the hypothesis referred to earlier. The following proof does not require this hypothesis.

To prove that Y implies X, assume Y true and X false. Then the proof is complete if we are led to a contradiction.

Commutative multiplication is assumed here, but the proof can easily be extended to a noncommutative number system. Unique division, except by zero, is also assumed.

A set, Γ_m, is built up of vectors $\gamma_{1,m}$, $\gamma_{2,m}$, $\cdots$, $\gamma_{N,m}$ which, together with all their linear combinations, will include all possible vectors orthogonal to all the columns of $\alpha_{,}''$. There appears to be no obvious way to write down the set Γ_m at one step, but it can be done by mathematical induction. Two sequences of sets of vectors are formed

$$A_0, A_1, \cdots, A_m,$$

$$\Gamma_0, \Gamma_1, \cdots, \Gamma_m.$$

A_k is the set of vectors consisting of the first k columns of $\alpha_{,}''$, and Γ_k is a set of vectors which, together with all their linear combinations, includes all the vectors orthogonal to all the vectors of A_k.

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* E. H. Moore, *Lectures on General Analysis.* Memoirs of the American Philosophical Society, vol. 1, 1935.

Since A_0 is a null set, the vectors of Γ_0 may be written down, by inspection, as the rows of Hilbert's unit matrix δ_I', namely,

$$\Gamma_0 = (\gamma_{1,0}', \gamma_{2,0}', \cdots, \gamma_{n,0}'),$$

where

$$\gamma_{1,0}' = 1, 0, 0, \cdots, 0, \qquad \gamma_{2,0}' = 0, 1, 0, \cdots, 0, \qquad \cdots.$$

The formation of $\Gamma_{(k+1)}$ from Γ_k is as follows. All the vectors of Γ_k are assumed orthogonal to all those of A_k. Now $A_{(k+1)}$ involves one new vector v, not in A_k. To see if all the vectors of Γ_k are accidentally orthogonal also to v_I, we consider the following N_k numbers:

$$Q_j = \gamma_{j,k}' v', \qquad (j = 1, 2, \cdots, N_k),$$

where N_k is the number of vectors $\gamma_{1,k}, \cdots, \gamma_{N,k}$ in Γ_k.

CASE 1. Now if all the numbers Q_j are zero, then all the vectors of Γ_k are orthogonal to all the vectors of $A_{(k+1)}$. This will happen, of course, if v, the $(k+1)$st column of α_I'' is linearly dependent on the preceding columns. It may, as far as we have yet shown, happen in other cases.

CASE 2. All the numbers Q_j are not zero, so that $Q_J \neq 0$, where J is some particular value of j. Let P denote the reciprocal of Q_J. In our number system such reciprocals are supposed to exist uniquely. As a tentative definition of $\Gamma_{(k+1)}$, consider

$$[\gamma_{j,k+1}'] \equiv [\gamma_{j,k}' - Q_j P \gamma_{J,k}'].$$

To test that all the vectors $\gamma_{j,(k+1)}'$ are orthogonal to all the vectors of $A_{(k+1)}$, note that it is obvious for the first k of the latter. In the case of v_I,

$$\gamma_{j,(b+1)}' v_I = \gamma_{j,b}' v_I - Q_j P \gamma_{J,b}' v_I$$
$$= Q_j - Q_j P Q_J = 0,$$

by the definition of P. In testing the linear independence of the vectors of $\Gamma_{(k+1)}$ as defined above, we find that one of them, $\gamma_{J,(k+1)}'$, is identically zero. That one must be deleted, and the others constitute the set $\Gamma_{(k+1)}$ as finally defined, so that the number of vectors in $\Gamma_{(k+1)}$ is one less than in Γ_k, or

$$N_{(k+1)} = N_k - 1.$$

The linear independence of the $N_k - 1$ vectors of $\Gamma_{(k+1)}$ follows directly from the linear independence of the vectors of Γ_k. Thus the rank of Γ_k is N_k, the same as the number of vectors in it. Let the rank of A_k be R_k. Then in Case 1,

$$N_{(k+1)} = N_k, \qquad R_{(k+1)} = R_k, \quad \text{or} \quad R_k + 1,$$

according as ν_i is or is not linearly dependent on the vectors of A_k. In Case 2,

$$N_{(k+1)} = N_k - 1, \qquad R_{(k+1)} = R_k + 1.$$

Therefore, in both cases,

$$N_{(k+1)} + R_{(k+1)} \geqq N_k + R_k.$$

When $k = 0$, $N_0 = n =$ the number of rows in α_i'' and $R_0 = 0$; so that $N_0 + R_0 = n$, and therefore $N_k + R_k \geqq n$.

Now prolong the sequence $A_0, \cdots, A_m$, by defining $A_{(m+1)}$ as the set of all columns of α_i'' and the vector η_i. Then $\Gamma_{(m+1)}$ is formed from Γ_m in the manner described for forming $\Gamma_{(k+1)}$ from Γ_k.

To obtain the contradiction, we are assuming Y true and X false. Since Y is true, all the vectors of Γ_m are orthogonal to all the vectors of $A_{(m+1)}$, so we have Case 1, where $\Gamma_{(m+1)} = \Gamma_m$. Therefore $N_{(m+1)} = N_m$.

Since X is false, the last vector of $A_{(m+1)}$ is not expressible linearly in terms of the vectors of A_m. Therefore $R_{(m+1)} = R_m + 1$, so that

$$N_{(m+1)} + R_{(m+1)} > N_m + R_m \geqq n.$$

Now the sets $A_{(m+1)}$ and $\Gamma_{(m+1)}$ are mutually orthogonal, and the vectors of those sets have only n elements. It is, therefore, impossible for the sum of their ranks to exceed n, and we have the desired contradiction.

Or, by extending the sequence of A's further, one could obtain as a contradiction a single set of vectors in n-space with rank $n + 1$. This, of course, could be reduced step by step to a set of vectors in one dimension of rank two.

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