ON EXTENDING A HOMEOMORPHISM BETWEEN TWO SUBSETS OF SPHERES*

BY H. M. GEHMAN

In two papers previously published,† the author has determined conditions under which a homeomorphism, or continuous (1-1) correspondence, between two plane point sets of a certain type can be extended to a homeomorphism between their planes. The two types of point set which have been considered are (a) a continuous curve, and (b) a closed bounded set, each component of which is a continuous curve, not more than a finite number of components being of diameter greater than any given positive number. In a recent paper,‡ Adkisson has determined, for case (a), conditions under which a homeomorphism between two subsets of spheres can be extended to a homeomorphism between the spheres. The object of this paper is to generalize Adkisson's results by proving a similar theorem for case (b). Finally it is shown how any theorem concerning the extension of a homeomorphism between plane sets yields a corresponding theorem for subsets of spheres, and conversely.

DEFINITION.§ An *E-set* is a closed proper subset of a sphere, each component of which is a continuous curve, not more than a finite number of components being of diameter greater than any given positive number.

THEOREM.‖ *Let M and M' be E-sets on the spheres S and S'*

---

* Presented to the Society, September 13, 1935.

† H. M. Gehman, *On extending a continuous* (1-1) *correspondence of two plane continuous curves to a correspondence of their planes*, Transactions of this Society, vol. 28 (1926), pp. 252–265, and H. M. Gehman, *On extending a continuous* (1-1) *correspondence* (*Second paper*), Transactions of this Society, vol. 31 (1929), pp. 241–252.

‡ V. W. Adkisson, *On extending a continuous* (1-1) *correspondence of continuous curves on a sphere*, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, vol. 27 (1934), pp. 5–9.

§ See Gehman, *Second paper*, p. 241. For other definitions, see papers previously cited.

‖ See Gehman, *Second paper*, Theorem 2, p. 244, and paragraph 2, p. 252.

respectively, and let T be a homeomorphism such that $T(M)=M'$. If $S-M$ and $S'-M'$ contain points x and x', respectively, such that T preserves sides in the same sense in the planes $S-x$ and $S'-x'$, then T can be extended to a homeomorphism U between the spheres S and S'. Conversely, if T can be extended to a homeomorphism between the spheres S and S', then T preserves sides in the same sense in the planes $S-x$ and $S'-x'$, where x is any point of $S-M$, and $x'=U(x)$.

By Theorem 2 of *Second paper*, we know that T can be extended to a homeomorphism V between $S-x$ and $S'-x'$. Let us define a correspondence U between S and S' as follows: $U(x)=x'$; for each point y of $S-x$, $U(y)=V(y)$. This correspondence is evidently (1-1). If M is a subset of S, and a point y of $S-x$ is a limit point of M, and hence of $M-Mx$, then, since V is continuous, the point $U(y)=V(y)$ is a limit point of $U(M-Mx)$ and hence of $U(M)$. If x is a limit point of M, then in the plane $S-x$ the set $M-Mx$ is unbounded. Hence $V(M-Mx)$ is unbounded in the plane $S'-x'$, and consequently the point $x'=U(x)$ of S' is a limit point of $V(M-Mx)$ $=U(M-Mx)$ and of $U(M)$. Hence the correspondence U preserves limit points. In the same way it may be shown that U^{-1} preserves limit points. Hence U is the required homeomorphism. The converse part of the theorem is obvious.

Since a sphere minus a point is topologically equivalent to a plane, and since by the argument used above, a homeomorphism between two such planes can be extended to a homeomorphism between the spheres containing them, it follows that any theorem concerning the extension of a homeomorphism between subsets of planes yields a theorem concerning the extension of a homeomorphism between certain subsets of spheres.

Similarly a plane plus a point at infinity is topologically equivalent to a sphere. Any homeomorphism between two such spheres under which the points at infinity correspond, defines a homeomorphism between the two planes. Hence any theorem concerning the extension of a homeomorphism between proper subsets of spheres yields a corresponding theorem for certain subsets of planes.

The above remarks also hold true if we consider the extension of a homeomorphism in the sense of Antoine.* From Theorem 2, p. 394, of the paper just cited, we can obtain a theorem for A-extending a homeomorphism between two subsets of spheres.

UNIVERSITY OF BUFFALO

A PROPERTY OF THE SOLUTIONS OF $t^2-du^2=4$

BY GORDON PALL

Let p be any odd prime not dividing d. The integral solutions t_i, u_i, $(i=0,\pm 1, \cdots)$,† of $t^2-du^2=4$ have the following property.

THEOREM. *Let $m+n=r+s$. Let v stand for t or u. Then $v_m+v_n\equiv v_r+v_s$ (mod p) if and only if the terms are congruent in pairs;‡ the same holds for each of*

$$v_m - v_n \equiv v_r - v_s, \quad v_m + v_n \equiv -(v_r + v_s), \quad v_m - v_n \equiv -(v_r - v_s).$$

For if $m+n$ is even and $v=u$, we can write $m=h+i$, $n=h-i$, $r=h+j$, $s=h-j$, whence

$$u_m + u_n = u_h t_i, \quad u_r + u_s = u_h t_j;$$

if $u_h=0$, then $u_m\equiv -u_n$; if $t_i\equiv t_j$, known conditions for two u's or t's to be congruent show that $u_m=u_r$ or u_s. The remaining cases are similar. If $m+n$ is odd, we transpose terms, and find with a little attention to parities $(u_i=-u_{-i},\ t_i=t_{-i})$ one or other of the former cases.

MCGILL UNIVERSITY

---

* H. M. Gehman, *On extending a correspondence in the sense of Antoine*, American Journal of Mathematics, vol. 51 (1929), pp. 385–396.

† For notations see, for example, Pall, Transactions of this Society, vol. 35 (1933), p. 501.

‡ That is, $v_m\equiv -v_n$, v_r, or v_s.