COROLLARY 1. The absolute minimum value of k is zero; this value is taken on if the midpoint of the line segments (g_t, g_t') and $(g, 1/\bar{g})$ coincide and is possible only for T an elliptic transformation.

PROOF. Substituting $m=-(\alpha\bar{\alpha}+\nu\bar{\nu})/(2\alpha\nu)$ into (2), we see that $k=0$ if $(a-\bar{a})/(2c)=-(\alpha\bar{\alpha}+\nu\bar{\nu})/(2\alpha\nu)$. Furthermore, we have $Q_0[-(\alpha\bar{\alpha}+\alpha\bar{\nu})/(2\alpha\nu)]>0$ for all G and all T of Fuchsian type, whereas $Q_0[(a-\bar{a})/(2c)]>0$ for T elliptic only.

REMARK 3. Changing (2) to trigonometric form, one finds the discriminant of the resulting quadratic in ρ to be

$$f(k) = 4(\alpha\nu e^{i\theta} + \bar{\alpha}\bar{\nu}e^{-i\theta})^2 - 16\alpha\bar{\alpha}\nu\bar{\nu}(1 - k^2).$$

This is a perfect square if and only if $k=1$ or 0; hence (2) is factorable rationally in terms of the coefficients of G in these two cases and only in them. The factors for $k=1$ are Q_5 and Q_6 of Theorem 1, and for $k=0$ they are immediate from (2).

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THE EQUATION $2^x-3^y=d$*

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1. *Introduction.* According to Dickson's *History of the Theory of Numbers*,† Leo Hebreus, or Levi Ben Gerson (1288–1344), proved that $3^m \pm 1 \neq 2^n$ if $m>2$, by showing that $3^m \pm 1$ has an odd prime factor. The problem had been proposed to him by Philipp von Vitry in the following form: All powers of 2 and 3 differ by more than unity except the pairs 1 and 2, 2 and 3, 3 and 4, 8 and 9. In 1923 an elegant short proof by Philip Franklin appeared in the American Mathematical Monthly.‡

In 1918 G. Polya§ published a very general theorem which, as was later pointed out by S. Sivasankaranarayana Pillai,|| proved as special cases that the equations

* Presented to the Society, October 26, 1935.

† Vol. 2, p. 731; see J. Carlebach, Dissertation, Heidelberg, 1909, pp. 62–64.

‡ Vol. 30 (1923), p. 81, problem 2927.

§ *Zur Arithmetische Untersuchung der Polynome*, Mathematische Zeitschrift, vol. 1 (1918), pp. 143–148.

|| Journal of the Indian Mathematical Society, vol. 19 (1931), pp. 1–11.

$$a^x - b^y = d, \tag{1}$$

$$2^x - 3^y = d, \tag{2}$$

where a and b are fixed positive integers and $d \neq 0$, have at most a finite number of solutions in positive integers (x, y). Moreover, Pillai gave an asymptotic formula for the finite number of solutions of the inequality $0 < a^x - b^y \leqq n$, where $\log a/\log b$ is not rational.

In this paper we shall prove that if $|d|$ is sufficiently great, equation (2), for fixed d, can have at most one solution, while equation (1) can have no more than nine solutions.

The general existence theorems tell us in particular that the inequality $|2^x - 3^y| \leqq n$ holds for only a finite number of pairs of positive integers (x, y). But they do not tell us the precise values of such pairs (x, y) nor exactly how many exist. We shall answer these questions for $n = 10$ by solving the six equations $2^x - 3^y = d$, $d = \pm 1, \pm 5, \pm 7$, since we may obviously exclude $d = 0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 10$. Similar methods apply to greater values, and we summarize some results for $n = 100$.

2. *Special Results.* If $2^x - 3^y = d$ and $x \geqq 3$, then $3^y \equiv -d \pmod 8$. But $3^n \equiv 1$ or $3 \pmod 8$. Hence $d \equiv 5$ or $d \equiv 7 \pmod 8$. Consequently for $d = 1, -5, -7$ there are no solutions of equation (1) such that $x \geqq 3$. We see therefore that these equations have only the solutions

$$2^2 - 3 = 1, \qquad 2^2 - 3^2 = -5, \qquad 2 - 3^2 = -7.$$

There remain the cases $d = -1, 5, 7$. Suppose $2^x - 3^y = -1$, so that $3^y \equiv 1 \pmod{2^x}$. But if $x > 2$, 3 belongs* to 2^{x-2} modulo 2^x, that is, 2^{x-2} is the least integer e such that $3^e \equiv 1$ modulo 2^x. Therefore $y \geqq 2^{x-2}$, if $x > 2$, and so

$$3^y - 1 \geqq 3^{2^{x-2}} - 1 \geqq 3^x - 1 > 2^x,$$

if $x \geqq 4$. Consequently $2^x - 3^y = -1$ is impossible if $x \geqq 4$. Thus the only solutions are

$$2 - 3 = -1 = 2^3 - 3^2.$$

Next consider the equation $2^x - 3^y = 5$. Obviously we have as two solutions $(x, y) = (3, 1)$ and $(5, 3)$. Suppose another solu-

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* See Dickson, *Introduction to the Theory of Numbers*, pp. 16, 19.

tion (x, y) exists satisfying $2^x=3^y+5$ with $y>3$, and consequently $x>5$. Taking congruences modulo 2^5, $3^y\equiv -5 \pmod{2^5}$. But $3^3=2^5-5\equiv -5 \pmod{2^5}$ and 3 belongs to 8 modulo 2^5. Therefore $y=3+8k$, where $k>0$. Now take congruences modulo 2^6; since $x>5$, $3^y\equiv -5 \pmod{2^6}$. But $3^3\not\equiv -5 \pmod{2^6}$ and 3 belongs to 16 (mod 2^6). Thus k cannot be even and it follows that $y=11+16k'$.

Finally we take congruences modulo 17, noting that 3 belongs to 16 (mod 17), so that $2^x=3^y+5\equiv -10+5\equiv -5$ (mod 17). But we can easily verify that no power of 2 is congruent to -5 (mod 17). Hence we have a contradiction.

Lastly consider $2^x-3^y=7$, where $y>3$, so that $x>2$. Taking congruences modulo 3 and then modulo 4, we see that both x and y are even. Therefore

$$1 \leqq 2^{x/2} - 3^{y/2} = 7/(2^{x/2} + 3^{y/2}) < 1,$$

which is a contradiction. Hence the only solution is $2^4-3^2=7$.

We may summarize by tabulating our results:

d	(x, y)	d	(x, y)
-1	(1, 1); (3, 2)	1	(2, 1)
-5	(2, 2)	5	(3, 1); (5, 3)
-7	(1, 2)	7	(4, 2)

3. *General Theorems*. We have now shown that when $x>5$, or $y>3$, then $|2^x-3^y|>10$. The author has verified in an unpublished paper that similar methods may be used to prove that if x and y are positive integers such that $x>8$ or $y>5$, then $|2^x-3^y|>100$. In no case were there found more than two solutions of the equation (2) for any fixed d with $|d|\leqq 100$. We conclude by proving that *if $|d|$ is sufficiently great, the equation $2^x-3^y=d$ cannot have more than one solution.*

We shall use the property proved by S. S. Pillai in his 1931 paper, previously mentioned, that given any $\delta>0$ and integers a and b such that $\log a/\log b$ is not rational, there exists an integer $x_1=x_1(\delta)$ such that for all $x>x_1$ and all positive (a^x-b^y),

$$0<a^x-b^y>a^{x(1-\delta)}.$$

Suppose that $2^x-3^y=2^X-3^Y=d$, $X>x$, $Y>y$. Then

$$2^X - 2^x = 3^Y - 3^y, \qquad 2^x(2^{X-x} - 1) = 3^y(3^{Y-y} - 1),$$
$$2^{X-x} \equiv 1 \pmod{3^y}, \qquad 3^{Y-y} \equiv 1 \pmod{2^x},$$
$$X - x \geqq 2\cdot 3^{y-1}, \quad X \geqq 2\cdot 3^{y-1}, \quad Y - y \geqq 2^{x-2}, \quad Y \geqq 2^{x-2},$$

if $x > 2$.

Consider first $d > 0$. Choose any positive $\delta < 1/2$ and let $x_1 = x_1(\delta)$, where for all $x > x_1$ and $0 < 2^x - 3^y$, we have $0 < 2^x - 3^y > 2^{x(1-\delta)} > 2^{x/2}$. Let us consider only the positive values of d greater than 2^{x_1+5}. Then

$$2^x > d > 2^{x_1+5}, \qquad x > x_1 + 5, \qquad X > x > x_1,$$
$$2^X = 3^Y + d > 3^Y \geqq 3^{2^{x-2}} > 2^{2^{x-2}}, \qquad X > 2^{x-2} > 2x,$$

since $x > 5$. Therefore $d = 2^X - 3^Y > 2^{X/2} > 2^x > d$, a contradiction.

Next, for $d < 0$, consider only values of d such that $|d| > 3^{y_1+2}$, where $y_1 = y_1(\delta)$, so that for all $y > y_1$ and $0 < 3^y - 2^x$, we have $0 < 3^y - 2^x > 3^{y(1-\delta)} > 3^{y/2}$. Hence, if (2) has two solutions,

$$3^y > |d| > 3^{y_1+2}, \qquad y > y_1 + 2, \qquad Y > y > y_1,$$
$$3^Y = 2^X - d > 2^X \geqq 2^{2\cdot 3^{y-1}} > 3^{3^{y-1}}, \qquad Y > 3^{y-1} > 2y,$$

since $y > 2$. Thus $-d = 3^Y - 2^X > 3^{Y/2} > 3^y > -d$, a contradiction.

In conclusion we can say of the general equation $a^x - b^y = d$, that if $|d|$ is sufficiently great, this equation can have at most nine solutions. This is a simple consequence of the theorem due to C. L. Siegel,* that $ax^n - by^n = k$ (fixed $n \geqq 3$) has at most one solution if $|ab|$ is sufficiently great. For if $|d|$ is large enough we can write equation (1) in the form of at least one of nine equations

$$a^i A u^3 - b^j B v^3 = d, \qquad (i, j = 0, 1, 2),$$

where $|AB|$ is so great that each of the nine equations must, by Siegel's theorem, have at most one solution.

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* Abhandlungen Akademie Berlin, 1929, Nr. 1, 70 pp.