CONVEX EXTENSION AND LINEAR INEQUALITIES*

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A few years ago, at the Des Moines meeting, it was my privilege to address the Society on the subject *Linear inequalities*. In its simplest form the problem there considered had to do with a system of conditions

$$\sum_{j=1}^{n} a_{ij}x_j > 0, \qquad (i = 1, 2, \cdots, m), \tag{1}$$

the coefficients a_{ij} being given real constants, and the x_j unknowns to be determined so as to satisfy the given conditions. The treatment was entirely analytic, and the aim was to develop a theory dictated by analogy with the theory of linear equations.

Today my purpose is to focus attention on geometric aspects of the theory, and in particular to show its close relationship to a certain geometric notion which in recent years has been useful in many investigations in analysis.

Let us consider an n-dimensional euclidean space $\mathfrak{U}$ of points $u \equiv (u_1, u_2, \cdots, u_n)$. A set of points in $\mathfrak{U}$ is said to be *convex* if the membership of two points $u^{(1)}$ and $u^{(2)}$ in the set implies the membership of all points on the line segment joining $u^{(1)}$ and $u^{(2)}$. A given set may or may not be convex, but any set $\mathfrak{M}$ may be *extended* so as to be convex by adjunction of the necessary points. The resulting set, which may be defined logically as the greatest common subset of all the convex sets which contain $\mathfrak{M}$, will be called the *convex extension* of $\mathfrak{M}$ and denoted by $C(\mathfrak{M})$.

This extended set, under various names and definitions, has been the subject of considerable study, and has been useful in many applications. One may refer to Minkowski's *Geometrie der Zahlen*, 1910; to Carathéodory's paper *Ueber der Variabilitätsbereich der Fourierschen Konstanten* in the Rendiconti del Circolo Matematico di Palermo (vol. 32 (1911)); or to the recent com-

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prehensive work by Bonnesen and Fenchel entitled *Theorie der Konvexen Körper*, 1934, which includes an extensive bibliography.*

Let us note some of the properties of the convex extension $C(\mathfrak{M})$, limiting ourselves for simplicity to the case in which the original set $\mathfrak{M}$ is closed and bounded. The convex extension is easily seen to be closed, bounded, and perfect if $\mathfrak{M}$ contains more than a single point. It has the same dimensionality as the set $\mathfrak{M}$, and if it is truly n-dimensional,† it possesses *inner* points. It possesses two further striking properties, rigorously established by Carathéodory, each of which is indeed characteristic of the convex extension.

The first characterizes the points of $C(\mathfrak{M})$ in terms of the planes ($(n-1)$-flats) in which they lie; and some preliminary definitions and remarks must precede its statement.

A plane ($(n-1)$-flat) in $\mathfrak{U}$ is the locus of points $u \equiv (u_1, u_2, \cdots, u_n)$ satisfying a linear equation

$$\pi(u) \equiv c_0 + \sum_{j=1}^{n} c_j u_j = 0,$$

in which not all the coefficients c are zero. Any such plane determines two *open half-spaces* in $\mathfrak{U}$, consisting respectively of the points u for which $\pi(u) > 0$ and $\pi(u) < 0$. Each of these half-spaces becomes *closed* upon the adjunction of the points of the bounding plane $\pi(u) = 0$. Relative to our given closed and bounded set $\mathfrak{M}$, the plane $\pi(u) = 0$ will be called:

(1) *a bounding plane* (*Schranke*), if the points of $\mathfrak{M}$ are all in the same one of the two *open* half-spaces determined by $\pi(u) = 0$;

* Reference should also be made to the following two very recent articles which of course do not appear in the bibliography of Bonnesen and Fenchel: *Elementare Theorie der konvexen Polyeder*, by Weyl in Commentarii Mathematici Helvetici, vol. 7 (1935); and *Integration of functions with values in a Banach space*, by Garrett Birkhoff in the Transactions of this Society, vol. 38 (1935).

As to terminology, the most commonly used name for what I have called the convex extension is the "konvexe Hülle" (or convex hull) of a set of points. This name, as it is used, seems to me not only inappropriate but indeed misleading. It might very fittingly be used to designate the aggregate of *boundary* points of the convex extension of a closed and bounded set.

† The term truly n-dimensional will be used to describe a set of points which is in an n-dimensional space but in no $(n-1)$-flat of that space.

(2) *a supporting plane* (*Stützebene*), if the points of $\mathfrak{M}$ are all in the same one of the *closed* half-spaces determined by $\pi(u)=0$, and at least one point of $\mathfrak{M}$ is in the plane $\pi(u)=0$;

(3) *a separating plane*, if each open half-space determined by $\pi(u)=0$ contains a point of $\mathfrak{M}$.

We can now state the first of Carathéodory's characteristic properties of $C(\mathfrak{M})$.

PROPERTY C1. The convex extension of $\mathfrak{M}$ consists of those points of $\mathfrak{U}$ through which pass no bounding planes of $\mathfrak{M}$. Furthermore, the *inner* points of the convex extension are those through which pass no supporting planes, and the *boundary* points are those through which pass supporting planes.

This characterization furnishes an ideal geometric representation for the study of linear inequalities. To apply it to our system

$$\sum_{j=1}^{n} a_{ij}x_j > 0, \qquad (i = 1, 2, \cdots, m), \tag{1}$$

we take for the set of points $\mathfrak{M}$, the m points

$$\mathfrak{M}: \qquad (a_{i1}, a_{i2}, \cdots, a_{in}), \qquad (i = 1, 2, \cdots, m),$$

and consider the relation of this point set $\mathfrak{M}$ to the planes

$$\sum_{j=1}^{n} c_j u_j = 0$$

through the origin.

If any one of these planes is a bounding plane of $\mathfrak{M}$, its coefficients $(c_1, c_2, \cdots, c_n)$ (or their negatives) constitute a solution of (1). And conversely, to every solution of (1) there corresponds a bounding plane of $\mathfrak{M}$ through the origin. Hence we have the following theorem.

THEOREM 1. *A necessary and sufficient condition for the existence of a solution of the system of inequalities* (1) *is that the origin* $(0, 0, \cdots, 0)$ *shall not belong to the convex extension of* $\mathfrak{M}$. *The solutions are the sets of coefficients* (*appropriately signed*) *of the bounding planes of* $\mathfrak{M}$ *through the origin.*

The distinction between inner points and boundary points of $C(\mathfrak{M})$ leads to an interpretation of the weaker system of inequalities considered by Minkowski

$$\sum_{j=1}^{n} a_{ij}x_j \geqq 0, \qquad (i = 1, 2, \cdots, m). \tag{2}$$

THEOREM 2. *A necessary and sufficient condition for the existence of a non-trivial* solution of the system of inequalities* (2) *is that the origin shall not be an inner point of the convex extension of* $\mathfrak{M}$. *The solutions are the sets of coefficients (appropriately signed) of the bounding planes and the supporting planes of* $\mathfrak{M}$ *through the origin.*

The two theorems just stated were obtained in essence by Miss R. W. Stokes in her dissertation published in the Transactions of this Society (1931). The geometric method she used enabled her also to study the character and representation of the solutions of the systems (1) and (2). The so-called fundamental solutions which play an important role in the analytic theory of Minkowski correspond to those supporting planes through the origin which contain $n-1$ points of $\mathfrak{M}$ which lie in no common $(n-2)$-flat. The *general* solution is a linear combination of the fundamental solutions, with positive coefficients in the case of (1) and non-negative coefficients in the case of (2).†

Recalling the manner in which Theorems 1 and 2 followed from a property of the convex extension $C(\mathfrak{M})$, we note that it is not essential that the number of points in $\mathfrak{M}$ be finite, that is, that the number of inequalities be finite as in (1) and (2). We may with equal ease consider systems

$$\sum_{j=1}^{n} f_j(p)x_j > 0, \qquad (p \text{ on the range } \mathfrak{P}), \tag{1p}$$

and

* Since the system (2) always admits the solution $x_j=0, (j=1, 2, \cdots, n)$, this will be called the *trivial* solution. The system (2) may admit other solutions for which the *equality* sign holds for every i. This will be the case if and only if the set $\mathfrak{M}$ lies in an $(n-1)$-flat through the origin. This is consistent with the theorem, since in that case the set $C(\mathfrak{M})$ has no inner points.

† The geometric method of approach has also been used by Haar, *Ueber lineare Ungleichungen*, Szeged Acta, sectio scientiarum mathematicarum, vol. 2 (1924); Fujiwara, *On the system of linear inequalities and linear integral inequality*, Proceedings of the Imperial Academy, vol. 4 (1928); Dines and McCoy, *On linear inequalities*, Transactions of the Royal Society of Canada, vol. 27 (1933).

$$\sum_{j=1}^{n} f_j(p)x_j \geqq 0, \qquad (p \text{ on the range } \mathfrak{P}), \tag{2p}$$

where $\mathfrak{P}$ is any class of elements p, the functions $f_j(p)$ are real-valued bounded functions on the range $\mathfrak{P}$, provided the set of points $\mathfrak{M}$ represented parametrically in the space $\mathfrak{U}$ by

$$\mathfrak{M}: \qquad u_1 = f_1(p),\ u_2 = f_2(p), \cdots, u_n = f_n(p), \qquad (p \text{ on } \mathfrak{P}),$$

is a closed set. The natural generalizations of Theorems 1 and 2 follow immediately.* However the following equivalent statement is more interesting from the point of view of analysis.

THEOREM 3. *A necessary and sufficient condition that every linear combination*

$$\sum_{j=1}^{n} c_j f_j(p), \qquad (p \text{ on range } \mathfrak{P}) \tag{3}$$

of the functions $f_j(p)$ shall change sign or vanish on $\mathfrak{P}$ is that the origin (0, 0, $\cdots$, 0) *shall belong to the convex extension of $\mathfrak{M}$. A necessary and sufficient condition that every linear combination* (3) *shall change sign on $\mathfrak{P}$ is that the origin shall be an inner point of the convex extension of $\mathfrak{M}$.*

So far our discussion has been based on the first of Carathéodory's characteristic properties of the convex extension. Let us now consider his second property.

PROPERTY C2. The convex extension of $\mathfrak{M}$ consists of those points of $\mathfrak{U}$ which can be the centroids of positive mass distributions (of total mass unity) at suitably chosen points of $\mathfrak{M}$.

Carathéodory showed that only a finite number (at most $n+1$) of points of $\mathfrak{M}$ are necessary thus to determine any point of $C(\mathfrak{M})$ as a centroid. Hence Property C2 can be expressed in analytic terms by saying that $C(\mathfrak{M})$ consists of those points of $\mathfrak{U}$ whose coordinates admit a representation of form

$$u_j = \sum_{i=1}^{r} \mu_i u_{ij}, \qquad (j = 1, 2, \cdots, n), \tag{4}$$

where

$$(u_{i1}, u_{i2}, \cdots, u_{in}), \qquad (i = 1, 2, \cdots, r), \tag{5}$$

* Dines and McCoy, loc. cit., p. 59.

is a set of points of $\mathfrak{M}$, and

$$\mu_i > 0, \qquad \sum_{i=1}^{r} \mu_i = 1.$$

The representation (4) of a given point of $C(\mathfrak{M})$ is of course not in general unique, and the possibility of variation is of interest, particularly in seeking a distinction between inner points and boundary points. The inner points are characterized by the fact that they admit such representation in terms of a *truly n-dimensional* subset of $\mathfrak{M}$. Remembering that the simplest set of this sort consists of $n+1$ points, one is tempted to associate that number with the representation of inner points. But such an association is erroneous. It is true that a point u which admits the representation (4) in terms of a truly n-dimensional subset of $n+1$ points of $\mathfrak{M}$ is an inner point of $C(\mathfrak{M})$, indeed of the n-dimensional simplex having these points as vertices. But not every inner point of $C(\mathfrak{M})$ necessarily admits such representation. For example, if $n=2$ and $\mathfrak{M}$ consists of the vertices of a square, the center of the square is an inner point of $C(\mathfrak{M})$ but it cannot be given the suggested representation in terms of *three* vertices. The following general statement can be made. An inner point of $C(\mathfrak{M})$ can be represented in terms of a truly n-dimensional subset which consists of not more than $2n$ points of $\mathfrak{M}$.*

For finite sets $\mathfrak{M}$, the inner points of $C(\mathfrak{M})$ are characterized in a slightly different manner in the following theorem, of which we shall see an application and an interesting analog for a certain type of infinite sets.

THEOREM 4. *If $\mathfrak{M}$ is a truly n-dimensional finite set of points, the inner points of the convex extension $C(\mathfrak{M})$ are precisely those for which all points of $\mathfrak{M}$ may be included in the set* (5) *of the representation* (4).

The proof, which is not so simple as one might expect, is quite similar to that of Theorem 12 in the paper by Dines and McCoy.

For infinite sets $\mathfrak{M}$, one is naturally led to attempt the representation of the points of $C(\mathfrak{M})$ by some of the infinite summation processes. Let us suppose, for instance, that $\mathfrak{M}$ consists of the points of a continuous arc defined parametrically by

* Dines and McCoy, loc. cit., pp. 61–63.

$$(6) \qquad u_1 = f_1(x),\ u_2 = f_2(x),\ \cdots,\ u_n = f_n(x), \qquad (a \leqq x \leqq b).$$

The obvious suggestion from analogy is that the points u of $C(\mathfrak{M})$ should be expressible in the form

$$(7) \qquad u_j = \int_a^b \mu(x) f_j(x) dx, \qquad (j = 1, 2, \cdots, n),$$

the function $\mu(x)$ being non-negative and such that

$$\int_a^b \mu(x) dx = 1.$$

But caution is necessary here. It is easily seen that every point $\bar{u}$ expressible in form (7) belongs to $C(\mathfrak{M})$. For otherwise there would be a bounding plane

$$\sum_{j=1}^n c_j(u_j - \bar{u}_j) = 0$$

through $\bar{u}$, such that

$$\sum_{j=1}^n c_j(f_j(x) - \bar{u}_j) > 0, \qquad (a \leqq x \leqq b);$$

whence, on multiplying both sides by $\mu(x)$ and integrating, one would have the contradiction

$$\sum_{j=1}^n c_j(\bar{u}_j - \bar{u}_j) > 0.$$

But the converse is not true, as the following simple example will show. Let the arc in question be the circular quadrant

$$u_1 = \cos \frac{\pi}{2} x, \qquad u_2 = \sin \frac{\pi}{2} x, \quad (0 \leqq x \leqq 1).$$

The point $(u_1, u_2) = (1, 0)$ belongs to $C(\mathfrak{M})$. But there is no non-negative function $\mu(x)$ such that

$$1 = \int_0^1 \mu(x) \cos \frac{\pi}{2} x\, dx, \qquad 0 = \int_0^1 \mu(x) \sin \frac{\pi}{2} x\, dx,$$

$$\int_0^1 \mu(x) dx = 1.$$

Obviously the difficulty here is due to the fact that (1, 0) is a boundary point of $C(\mathfrak{M})$. If we consider only *inner* points, the following interesting analog of Theorem 4 is valid.

THEOREM 5.* *If the continuous arc* (6) *does not lie in an* $(n-1)$-*flat, the inner points of its convex extension are precisely those points u whose coordinates can be expressed in the form*

$$u_j = \int_a^b \mu(x) f_j(x) dx, \qquad (j = 1, 2, \cdots, n), \tag{8}$$

where $\mu(x)$ *is continuous, and*

$$\int_a^b \mu(x) dx = 1, \qquad \mu(x) > 0, \qquad (a \leqq x \leqq b).$$

But in view of the nature of the problem, one can hardly overlook the fact that the Stieltjes integral is the ideal means for representing points of the convex extension. Relative to the continuous arc (6) we have the following theorem, the first part of which is due to F. Riesz.†

THEOREM 6. *The convex extension of the continuous arc represented parametrically by* (6) *consists of those points u which admit a representation*

$$u_j = \int_a^b f_j(x) d\alpha(x), \qquad (j = 1, 2, \cdots, n), \tag{9}$$

where $\alpha(x)$ *is a monotonic non-decreasing function such that*

$$\int_a^b d\alpha(x) = 1. \tag{10}$$

If the arc does not lie in an $(n-1)$-*flat, the inner points of the convex extension are those which admit the representation* (9) *with* $\alpha(x)$ *a monotonic increasing function satisfying* (10).

* Proved by Schoenberg, this Bulletin, vol. 39 (1933). It appears, however, that Schoenberg's hypothesis that the functions $f_j(x)$ be linearly independent should be replaced by the more restrictive condition that no linear combination of these functions be constant, which is equivalent to our hypothesis.

Another proof may be found in Fenchel's *Geschlossene Raumkurven mit vorgeschriebenem Tangentenbild*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 40 (1930).

† Annales de l'École Normale, vol. 28, pp. 56–57.

The proof, in view of our previous discussion, is quite simple. From well known properties of the Stieltjes integral it follows that the finite sum in formula (4), which represents any point of the convex extension, can be replaced by a Stieltjes integral of form (9) with $\alpha(x)$ satisfying (10). It suffices to take for $\alpha(x)$ a step function with jumps of magnitude μ_i at suitably chosen points.

Conversely, if a point $\bar{u}$ admits the representation (9) with $\alpha(x)$ monotonic non-decreasing and satisfying (10), the point belongs to the convex extension. For otherwise there would pass through it a bounding plane

$$\sum_{j=1}^{n} c_j(u_j - \bar{u}_j) = 0$$

such that

$$\sum_{j=1}^{n} c_j(f_j(x) - \bar{u}_j) > 0, \qquad (a \leqq x \leqq b). \tag{11}$$

If now we denote the left side of (11) by $F(x)$, so that

$$F(x) \equiv \sum_{j=1}^{n} c_j(f_j(x) - \bar{u}_j), \qquad (a \leqq x \leqq b),$$

we obtain a contradiction in the Stieltjes integral

$$\int_a^b F(x)d\alpha(x).$$

Since $F(x)$ is positive, this integral must be positive, while its evaluation by use of (9) (with $u = \bar{u}$) and (10) shows it to be zero.

To prove the second part of the theorem, we observe that formula (8), which will represent any *inner* point of the convex extension, can be thrown into the form (9) by taking

$$\alpha(x) = \int_a^x \mu(x)dx, \qquad (a \leqq x \leqq b).$$

Since $\mu(x)$ is positive and continuous, $\alpha(x)$ will be monotonic increasing, and it will satisfy (10) in view of the analogous condition on $\mu(x)$.

Conversely, if a point $\bar{u}$ admits a representation (9) with $\alpha(x)$ monotonic increasing and satisfying (10), then it must be an *inner* point of the convex extension. For if it were a boundary point there would pass through it a supporting plane

$$\sum_{j=1}^{n} c_j(u_j - \bar{u}_j) = 0$$

such that

$$F(x) \equiv \sum_{j=1}^{n} c_j(f_j(x) - \bar{u}_j) \geqq 0, \quad (a \leqq x \leqq b),$$

the *inequality* certainly holding for at least one value of x since the arc cannot lie in the supporting $(n-1)$-flat. If now we consider the Stieltjes integral of $F(x)d\alpha(x)$, a simple argument based on the properties of $F(x)$ and $\alpha(x)$ leads to the contradictory conclusions that this integral must be both positive and zero. The contradiction completes the proof of the theorem.

We interpreted the first Carathéodory property of the convex extension in terms of linear inequalities. The second property admits an even more obvious interpretation in terms of linear equations. From our discussion the following conclusions result.*

THEOREM 7. *A necessary and sufficient condition that the system of n linear homogeneous equations in m unknowns*

$$\sum_{i=1}^{m} a_{ij}\mu_i = 0, \qquad (j = 1, 2, \cdots, n), \tag{12}$$

admit a non-negative (and non-trivial) solution $(\mu_1, \mu_2, \cdots, \mu_m)$ *is that, in an n-dimensional euclidean space containing the set of points*

$$\mathfrak{M}: \qquad (a_{i1}, a_{i2}, \cdots, a_{in}), \qquad (i = 1, 2, \cdots, m),$$

the origin shall belong to the convex extension of that set. If the set of points $\mathfrak{M}$ *is truly n-dimensional, a necessary and sufficient condition that the system* (12) *admit a positive solution* $(\mu_1, \mu_2, \cdots, \mu_m)$ *is that the origin be an inner point of the convex extension of* $\mathfrak{M}$.

* For simplicity we limit our statements to the special case of homogeneous equations. The alterations to be made in the non-homogeneous case will be obvious.

THEOREM 8. *A necessary and sufficient condition that the system of Stieltjes integral equations*

$$\int_a^b f_j(x)d\alpha(x) = 0, \qquad (j = 1, 2, \cdots, n), \tag{13}$$

in which the functions $f_j(x)$ are continuous, admit a monotonic non-decreasing (and non-trivial) solution $\alpha(x)$ is that the origin belong to the convex extension of the arc

$$u_1 = f_1(x),\ u_2 = f_2(x), \cdots, u_n = f_n(x), \qquad (a \leqq x \leqq b).$$

If this arc does not lie in an $(n-1)$-flat, a necessary and sufficient condition that the system (13) admit a monotonic increasing solution $\alpha(x)$ is that the origin be an inner point of the convex extension.

And this last condition is also necessary and sufficient for the existence of a positive continuous solution $\mu(x)$ of the system of integral equations

$$\int_a^b f_j(x)\mu(x)dx = 0, \quad (j = 1, 2, \cdots, n).$$

In our discussion we have noted analytic interpretations of each of Carathéodory's two characteristic properties of the convex extension. The equivalence of these two properties of course implies the logical equivalence of the analytic interpretations. Each instance of such equivalence yields a purely analytic theorem. Let us note a few.

THEOREM 9. *A necessary and sufficient condition that the system of linear inequalities*

$$\sum_{j=1}^{n} a_{ij}x_j > 0, \qquad (i = 1, 2, \cdots, m),$$

admit a solution $(x_1, x_2, \cdots, x_n)$ is that the adjoint system of linear equations

$$\sum_{i=1}^{m} a_{ij}y_i = 0, \qquad (j = 1, 2, \cdots, n), \tag{14}$$

admit no non-negative solution $(y_1, y_2, \cdots, y_m)$ other than the trivial zero solution.

A necessary and sufficient condition that the system

$$\sum_{j=1}^{n} a_{ij}x_j \geqq 0, \qquad (i = 1, 2, \cdots, m),$$

admit a solution $(x_1, x_2, \cdots, x_n)$ *which does not annul all the left members, is that the adjoint system of linear equations* (14) *admit no positive solution* $(y_1, y_2, \cdots, y_m)$.

These two results were first stated by me in approximately this form in a note in the Annals of Mathematics in 1926. However, W. B. Carver had essentially obtained the first in a paper in the Annals of 1922. And, unfortunately unknown to either of us, Stiemke had obtained the essence of both in the Mathematische Annalen of 1915. All of the purely analytic proofs were quite complicated.

From our results concerning the convex extension of a continuous arc we draw the following conclusions.

THEOREM 10. *If the n functions* $f_j(x)$ *are continuous and linearly independent on the interval* $(a \leqq x \leqq b)$, *a necessary and sufficient condition that the system of Stieltjes integral equations*

$$\int_a^b f_j(x)d\alpha(x) = 0, \qquad (j = 1, 2, \cdots, n), \tag{15}$$

admit a monotonic non-decreasing solution $\alpha(x)$ (*other than a constant*) *is that every linear combination*

$$\sum_{j=1}^{n} c_j f_j(x) \tag{16}$$

shall vanish somewhere on the interval $(a \leqq x \leqq b)$.

A necessary and sufficient condition that the system (15) *admit a monotonic increasing solution* $\alpha(x)$ *is that every linear combination* (16) *shall change sign on* $(a \leqq x \leqq b)$.

The last condition is likewise necessary and sufficient for the existence of a positive continuous solution $\mu(x)$ *of the system*

$$\int_a^b f_j(x)\mu(x)dx = 0, \qquad (j = 1, 2, \cdots, n).$$

The third part of this theorem I stated and proved in the Transactions of 1928, by what now appears to be a very cumbersome method. It was obtained by the present simple argument by Schoenberg in the Bulletin of 1933. This type of argument seems first to have been applied to problems of this sort by Kakeya and Fujiwara in various papers in the Japanese journals between 1914 and 1930, though unfortunately some inaccuracies mar their results.

The theorems which we have just obtained may perhaps be described in a general way as *matrix-free* theorems concerning adjoint systems of linear conditions. Two adjoint systems arise from the same matrix. The properties of the matrix determine the nature of the solutions of each system. But once the characterization has been established, the matrix may be eliminated from consideration, and there results a relationship between the natures of the solutions of the adjoint systems.

In our study, the matrix has been interpreted as a set of points and we have been interested in the relationship of its convex extension to the origin of coordinates. Through the two Carathéodory properties this relationship is translated into equivalent properties of the two adjoint linear systems.

Of course the methods and results we have described invite extension and generalization in various directions. We have seen that no particular difficulty is presented by the *number* of conditions in one of the adjoint systems, the one associated with the number of points in the set $\mathfrak{M}$. However, such is not the case with the other system, in which the number of conditions coincides with the dimensionality n of the euclidean space $\mathfrak{U}$. Special instances in which the number of conditions in this system is infinite have been considered by F. Riesz, Schoenberg, McCoy, and myself. But a most tempting generalization would be a theory of the convex extension of a set of points in a space more general than the ordinary euclidean n-space. Contributions in this direction have been made by H. Kneser and Steinitz for a projective space, Keller for Hilbert space, Ascoli and Mazur for general linear spaces, Menger for general metric spaces, and Whitehead for the geometry of paths. But time does not permit a consideration of any of these on this occasion. I must close by thanking you most sincerely for the privilege of addressing you, and for your courteous attention.

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