second series is identically zero, so the inversion formula does not give an actual solution. Under these circumstances we are forced to leave the question of the completeness of S_1+1 in $C[0, 1]$ unanswered.*

YALE UNIVERSITY AND MASSACHUSETTS INSTITUTE OF TECHNOLOGY

GROUPS OF MOTIONS IN CONFORMALLY FLAT SPACES

BY JACK LEVINE

1. *Introduction.* In this paper we consider the problem of determining the conditions which a conformally flat space must satisfy in order that it may admit a group of motions. These conditions are expressed in Theorem 1. Conformally flat spaces admitting simply transitive groups of motions are considered in the last section. All summations are from 1 through n unless otherwise indicated.

2. *Killing's Equations.* The equations for determining the possible existence of groups of motions in a metric space are known as Killing's equations and are given by†

$$\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial \xi^k}{\partial x^j} + g_{jk}\frac{\partial \xi^k}{\partial x^i} = 0. \tag{1}$$

If V_n is conformally flat, there exists a coordinate system in which $g_{ij}=e_i\delta_j^i h^2$, where $e_i=\pm 1$. In this coordinate system (1) reduce to

$$e_i\frac{\partial \xi^i}{\partial x^j} + e_j\frac{\partial \xi^j}{\partial x^i} = 0, \qquad (i \neq j,\ i, j \text{ not summed}), \tag{2}$$

$$\xi^k\frac{\partial H}{\partial x^k} + \frac{\partial \xi^i}{\partial x^i} = 0, \qquad (i \text{ not summed},\ H = \log h). \tag{3}$$

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* The completeness of $1+S(\beta+1, \beta, \lambda)$ in $C[0, 1]$ is proved for $-1<\beta\leqq 2$ in a paper to appear in the Annals of Mathematics.

† L. P. Eisenhart, *Riemannian Geometry*, p. 234.

From (2) we obtain

$$e_i \frac{\partial^2 \xi^i}{\partial x^j \partial x^k} = - e_j \frac{\partial^2 \xi^j}{\partial x^i \partial x^k} = e_k \frac{\partial^2 \xi^k}{\partial x^i \partial x^j} = - e_i \frac{\partial^2 \xi^i}{\partial x^j \partial x^k},$$

$(i, j, k \neq$, no summing),

which gives

$$\frac{\partial^2 \xi^i}{\partial x^j \partial x^k} = 0, \qquad (i, j, k \neq). \tag{4}$$

From (3) we have

$$\frac{\partial \xi^i}{\partial x^i} = \frac{\partial \xi^j}{\partial x^j} \equiv \tau, \qquad (i, j \text{ not summed}). \tag{5}$$

From (2) follow

$$e_i \frac{\partial^2 \tau}{\partial x^j \partial x^j} + e_j \frac{\partial^2 \tau}{\partial x^i \partial x^i} = 0,$$

$$e_j \frac{\partial^2 \tau}{\partial x^k \partial x^k} + e_k \frac{\partial^2 \tau}{\partial x^j \partial x^j} = 0,$$

$$e_k \frac{\partial^2 \tau}{\partial x^i \partial x^i} + e_i \frac{\partial^2 \tau}{\partial x^k \partial x^k} = 0, \qquad (i, j, k \text{ not summed}),$$

so that $\partial^2 \tau / \partial x^i \partial x^i = 0$, ($i$ not summed), and by (4) we get $\partial^2 \tau / \partial x^i \partial x^j = 0$, so finally we have

$$\tau = \frac{\partial \xi^i}{\partial x^i} = a_0 + a_j x^j, \qquad (i \text{ not summed}), \tag{6}$$

where the a's are constants. The general solution ξ^i of (2) and (5) is found to be

$$\xi^i = b^i + a_0 x^i + x^i a_j x^j - \frac{1}{2} a_i e_i e_j (x^j)^2 + b_j^i x^j. \tag{7}$$

The a's and b's are arbitrary, with $e_i b_j^i + e_j b_i^j = 0$, ($i$, j not summed). The group generated by the ξ^i of (7) is the general conformal group of $(n+1)(n+2)/2$ parameters.* In order to define a group of motions, the ξ^i must satisfy the further conditions (3).

* S. Lie, *Theorie der Transformationsgruppen*, vol. 3, pp. 334, 347.

If we substitute the value of ξ^i as given by (7) into (3), we tain an equation which can be written as

$$\sum_{i=1}^{N} A_i u^i = 0, \qquad (N = (n+1)(n+2)/2), \tag{8}$$

where the A_i represent the N constants in the expression for ξ^i and u^i are functions of the x's. From (8) we obtain an infinite sequence of equations

$$\sum_{i=1}^{N} A_i \frac{\partial^t u^i}{\partial x^{a_1} \cdots \partial x^{a_t}} = 0, \qquad (t = 0, 1, 2, \cdots), \tag{9}$$

which must be identically satisfied in the x's. The function H being assumed analytic in a certain domain of the variables x, we may express the u^i in the form

$$u^i = u_0{}^i + u_j{}^i x^j + \frac{1}{2!} u^i_{jk} x^j x^k + \cdots,$$

and substituting for u^i in (9), we see that

$$\sum_{i=1}^{N} A_i u^i_{a_1 \cdots a_t} = 0, \qquad (t = 0, 1, \cdots). \tag{10}$$

Hence a necessary and sufficient condition for the existence of non-zero solutions for A_i is that the rank of the matrix

$$\left\| u^i_0,\ u^i_j,\ u^i_{jk},\ \cdots \right\|$$

be $\leqq N-1$. Since this condition must hold for every point in the domain of analyticity, we can replace the above matrix by

$$\left\| u^i,\ \frac{\partial u^i}{\partial x^j},\ \frac{\partial^2 u^i}{\partial x^j \partial x^k},\ \cdots \right\|.$$

This matrix in turn can be replaced by the finite matrix

$$\left\| u^i,\ \frac{\partial u^i}{\partial x^j},\ \cdots,\ \frac{\partial^{N-1} u^i}{\partial x^{a_1} \cdots \partial x^{a_{N-1}}} \right\|, \tag{11}$$

since we cannot have more than N independent equations (10).*

* Equations of the type (8) have been considered by M. S. Knebelman, who has obtained necessary and sufficient conditions for the existence of constant solutions A.

The u^i involve derivatives of H and the condition on the rank of the matrix (11) gives us the required restrictions on H. We state this in the following theorem.

THEOREM 1. *Given a function H defining a conformally flat space, a necessary and sufficient condition that this space admit a group of motions is that the rank of the matrix* (11) *be* $\leqq N-1$, *where* $N=(n+1)(n+2)/2$. *If the rank is* $N-r$, *the space admits a* G_r *of motions.*

We shall find the conditions for a conformally flat space with $h^2=1/f(r^2)$ to admit a group of motions, where

$$r^2=\sum e_i(x^i)^2.$$

For this case (3) becomes

$$\frac{f'}{f}=\frac{a_0+a_ix^i}{\sum e_ib^ix^i+(2a_0+a_ix^i)r^2/2}.$$

In order for the right member of the above equation to be a function of r^2, we must have

$$a_i = b^i = 0, \quad \text{or} \quad a_0 = 0, \quad e_ib^i = ca_i, \qquad (i \text{ not summed}).$$

For the second case we find that $f(r^2)$ must be of the form

$$f(r^2) = (\alpha r^2+\beta)^2, \qquad (\alpha, \beta \text{ const.}), \tag{12}$$

that is, the space is of constant curvature.* For the first case f must be of the form $f(r^2)=\alpha r^2$, where α is a constant. This proves the following theorem.

THEOREM 2. *The only metric spaces with quadratic form* $[1/f(r^2)]\sum e_i(dx^i)^2$ *which admit a group of motions are spaces of constant curvature, where f has the form* (12), *and spaces with the quadratic form* $(1/\alpha r^2)\sum e_i(dx^i)^2$. *The r^2 has the value* $r^2=\sum e_i(x^i)^2$.

It can be shown that if $H=\alpha+\alpha_ix^i$, the corresponding space admits a group of motions of at least $n-1$ parameters, and if $H=\alpha+(1/2)\alpha_{ij}x^ix^j$, the corresponding space admits a group of motions of at least n parameters, the α's being arbitrary. This is shown by a consideration of the rank of (11) for these choices of H.

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* *Riemannian Geometry*, p. 85.

3. *Simply Transitive Groups.* If a motion in V_n is to be a translation, we must have $e^{2H}\sum e_i(\xi^i)^2=c$, where c is a constant. If we calculate $\partial H/\partial x^i$ from this relation and substitute in (3), it is found that $\tau=0$, and hence a necessary condition for a translation is that ξ^i be of the form $\xi^i=b^i+b_j^i x^j$. Suppose V_n admits a simply transitive group G_n of translations. Then from (3) and the fact that $\tau=0$, it is easily seen that H is constant, that is, the V_n is a flat space. This gives us the following theorem.

THEOREM 3. *If a conformally flat space admits a simply transitive group of translations, the space is flat.*

If the e_i are all the same sign, it is easily shown that the group must be Abelian.

A simply transitive group G_n being given, we consider the conditions under which there exists a conformally flat space admitting G_n as a group of motions. Necessary and sufficient conditions are given by

$$\frac{\partial H}{\partial x^i} = L_i, \tag{13}$$

$$e_i L^i_{jk} + e_j L^j_{ik} = 0, \qquad (i, j \text{ not summed, } i \neq j), \tag{14}$$

where

$$L^i_{jk} = -\,\xi_{a|k}\frac{\partial \xi^i_{a|}}{\partial x^j}, \qquad L_i = L^1_{1i} = \cdots = L^n_{ni}, \tag{15}$$

$$(\xi_{a|k}\xi^k_{b|} = \delta^a_b,\ \xi_{a|i}\xi^j_{a|} = \delta^j_i).$$

From (14) and (15) it follows that (13) are completely integrable. Now if ξ^i satisfies (14) and (15) we have seen it must be of the form (7), and hence we have the following theorem.

THEOREM 4. *A given simply transitive group will be a group of motions of a conformally flat space if and only if the group is a sub-group of the general conformal group. The determination of the space depends on one arbitrary constant.*

NORTH CAROLINA STATE COLLEGE