LIE GROUPS SIMPLY ISOMORPHIC
WITH NO LINEAR GROUP[†]

BY GARRETT BIRKHOFF

1. *Introduction.* One of the most interesting conjectures concerning finite continuous groups is the conjecture that every Lie group is topologically isomorphic with a group of matrices. The proof of this conjecture, even in the small, would establish the truth of the famous conjecture that every Lie group nucleus (or *germ*) is a piece of a Lie group.[‡] This makes it of interest to know that there exist Lie groups in the large, simply isomorphic even in the purely algebraic sense—and a fortiori topologically isomorphic in the large—with no group of matrices. It is to the proof of this fact that the present note is devoted.

2. *The Basic Lemma.* The proof ultimately rests on the following lemma.

LEMMA 1. *Let* Γ *be any group of linear transformations. Suppose* Γ *contains elements* S *and* T *whose commutator* $R = S^{-1}T^{-1}ST$ *is of prime order* p, *and satisfies* $SR = RS$, $TR = RT$. *Then* Γ *is of degree at least* p.

By the *degree* of Γ is meant the *dimensions*—that is, the maximum number of linearly independent elements—of the linear space Σ on which Γ operates. The vector elements of Σ will be written $x, y, z, \cdots$.

PROOF. Since R is of order p, Σ contains a vector x such that $Rx = \alpha x$, where α is a primitive pth root of unity.[§] Now let Λ denote the linear subspace of *all* vectors $x \epsilon \Sigma$ satisfying $Rx = \alpha x$. If $x \epsilon \Lambda$, then

$$R(Sx) = S(Rx) = S(\alpha x) = \alpha(Sx).$$

[‡] For a fuller description of the status of these conjectures, see [3], especially §17. See also [1], p. 24, middle.

Numbers in square brackets refer to the Bibliography at the end of this paper.

[§] This follows from [4], Theorem 125.

That is, $Sx\epsilon\Lambda$, showing that S, and likewise T, transforms Λ into itself.

But now observe that *within* Λ, $S^{-1}T^{-1}ST = \alpha I$ (where I denotes the identical matrix), whence $T^{-1}ST = S$. But $T^{-1}ST$ and S have (see [4], Theorem 126) the same characteristic roots; hence so do S and αS, as linear transformations of Λ. Moreover since S is non-singular, its characteristic roots are not zero; hence it has at least p characteristic roots

$$\mu_0, \ \alpha\mu_0, \ \cdots, \ \alpha^{p-1}\mu_0, \qquad\qquad (\mu_0 \neq 0).$$

It follows that Λ, and therefore Σ, has at least p dimensions.

3. *The Main Theorem.* With the help of this lemma, one can easily exhibit a Lie group† simply isomorphic with no linear group. Let G_3 denote the group of all matrices

$$M(x, y, z) \ = \ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

And let N denote the discrete normal subgroup of matrices $M(0, 0, n)$, n any integer. Finally, let $G_3^* = G_3/N$.

THEOREM 1. G_3^* *is a Lie group simply isomorphic with no linear group.*

PROOF. That G_3^* is a Lie group can be verified directly, and also follows from general principles (see [1], p. 12). It remains to show that it is simply isomorphic with no linear group. But let p be any prime; then the images S of $M(1, 0, 0)$ and T of $M(0, 1/p, 0)$ both commute with $R = S^{-1}T^{-1}ST$, which is the image of $M(0, 0, 1/p)$ in G_3, while R is of order p. Hence by Lemma 1 any linear group simply isomorphic with G_3^* would have to be of degree at least p, and since p can be *any* prime, this is absurd.

COROLLARY. *No group having G_3^* as a subgroup is simply isomorphic with a linear group.*

REMARK 1. Since G_3 was a linear group, we see that there can

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† By a *Lie group* we shall mean a *connected finite continuous group* in the usual sense; *mixed* groups will be excluded. See [1], p. 7.

exist no construction yielding, from representations of a given group as a linear group, non-trivial representations of its homomorphic images as linear groups.

REMARK 2. As all the two-parameter Lie groups are known ([1], p. 25), and can be realized in the large as linear groups, G_3^* is the simplest example of a group isomorphic in the large with no linear group.

4. *Additional Definitions.* In §§4–6 we shall show how to find whole families of Lie groups simply isomorphic with no linear group. But in order to do this, we shall need to recall some familiar parts of group theory.

By the *central* of a Lie group G is meant the set of all elements of G commuting with all other elements; this is a topologically closed and invariant subgroup of G.

Every Lie group G has a *Lie algebra* A, defined from the set of the infinitesimal generators U, V, $\cdots$ of G by regarding commutation (the taking of Poisson brackets) as a non-associative multiplication. Two Lie groups are locally isomorphic if and only if they have isomorphic Lie algebras ([2], [32.2]).

We shall call G *hypercentral* if and only if the associated Lie algebra is *nilpotent*, in the sense that all products of sufficient length r reduce to zero. This terminology is justified since G is hypercentral if and only if, denoting by Z_1 the subgroup of G consisting of the identity 0, and by Z_{k+1}/Z_k inductively the central of G/Z_k, we have $Z_r = G$. (Actually, Z_k contains the invariant sub-algebra A_{r-k} of A consisting of linear combinations of products of lengths $\geqq (r-k)$.)

REMARK. Any Lie group of *triangular* matrices $A = \|a_{ij}\|$ satisfying $a_{ii} = 1$ and $a_{ij} = 0$ if $i > j$ is hypercentral. All Poisson brackets of length exceeding its degree vanish.

5. *A Theorem of Linear Groups.* We shall now prove a result of some intrinsic interest,† which will be used in the sequel.

THEOREM 2. *Let U, V, W be any three matrices which satisfy $UV - VU = W$, $UW = WU$, $VW = WV$. Then the matrices $M(\lambda, \lambda', \lambda'') = \exp(\lambda U + \lambda' V + \lambda'' W)$ form a group H, topologically isomorphic in the large with G_3.*

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† Theorem 2 is especially interesting as U, V, W have the formal properties of the position, momentum, and scalar operators p, q, $qp - pq$ of quantum mechanics.

EXPLANATION. As is customary,[†] we use the notation

$$\exp X = I + X + \frac{1}{2!} X^2 + \frac{1}{3!} X^3 + \cdots .$$

PROOF. Since for any X, $[\exp X] \circ [\exp (-X)] = I$, and since in H one can prove the special identity

(1)
$$\begin{aligned} M(\lambda, \lambda', \lambda'') \circ M(\mu, \mu', \mu'') \\ = M(\lambda + \mu, \lambda' + \mu', \lambda'' + \mu'' - \lambda'\mu), \end{aligned}$$

clearly H is a group. Further, (1) exhibits a continuous homomorphism[‡] of G_3 into H, since (1) expresses the rule for matrix multiplication in G_3.

We shall show that this is a topological isomorphism, in other words, that if $M(\lambda_n, \lambda_n', \lambda_n'') \equiv M_n \to M(\lambda, \lambda', \lambda'')$, then $\lambda_n \to \lambda$, $\lambda_n' \to \lambda'$, and $\lambda_n'' \to \lambda''$. By group-homogeneity, it is sufficient to prove this for the special case $\lambda = \lambda' = \lambda'' = 0$, $M(\lambda, \lambda', \lambda'') = I$. This is what we shall do.

The kernel of the demonstration is a well known theorem of Lie ([2], p. 163) which shows that one can so transform the matrices of H that the generating matrices $U = \|u_{ij}\|$ and $V = \|v_{ij}\|$ are *triangular*, that is, satisfy $u_{ij} = v_{ij} = 0$, if $i > j$. But now computing the matrix $W = UV - VU = \|w_{ij}\|$, we see that $w_{ij} = 0$ unless $(j - i) \geq 1$, while it is easily verified to be a general property of matrix multiplication, that if Y and Z are any two matrices such that $y_{ij} = 0$ unless $(j - i) \geq \alpha_1$, and $z_{ij} = 0$ unless $(j - i) \geq \alpha_2$, and $X = YZ$, then $x_{ij} = 0$ unless $(j - i) \geq \alpha_1 + \alpha_2$. It is a corollary that the non-zero coefficients of W nearest the principal diagonal, that is, minimizing $(j - i)$ subject to $w_{ij} \neq 0$, appear as $\lambda'' w_{ij}$ in $\exp(\lambda'' W)$, and so the subgroup of the $\exp(\lambda'' W)$ is topologically isomorphic in the large with the translation-group $x \to x + t$.

Unless $\lambda_n \to 0$, we can, taking a suitable subsequence, assume without loss of generality that $1/\lambda_n = \bar{\lambda}_n \to \bar{\lambda}$. But irrespective of n,

$$M_n^{-1} \circ \exp(-\bar{\lambda}_n V) \circ M_n \circ \exp(\bar{\lambda}_n V) = \exp W \neq I;$$

† See J. von Neumann, Mathematische Zeitschrift, vol. 30 (1929), pp. 3–42.

‡ For the distinction between continuous homomorphisms and topological isomorphisms see H. Freudenthal, *Einige Sätze über topologische Gruppen*, Annals of Mathematics, vol. 37 (1936), p. 46.

while on the other hand, $M_n \to I$, exp $(\bar{\lambda}_n V) \to$ exp $(\bar{\lambda} V)$, and $I^{-1} \circ$ exp $(-\bar{\lambda} V) \circ I \circ$ exp $(\bar{\lambda} V) = I$. Since group multiplication and passage to the inverse are continuous operations, this is absurd.

This proves $\lambda_n \to 0$; similarly $\lambda_n' \to 0$. Hence it follows that exp $(\lambda_n'' W) \to I$, whence by the preceding paragraph $\lambda_n'' \to 0$, which completes the proof.

REMARK. It is a corollary of §5 alone that G_3^* is topologically isomorphic in the large with no linear group. (For its generators satisfy the hypotheses of Theorem 2.)

6. *The Extended Theorem.* Using Theorems 1 and 2, one can exhibit without difficulty a whole class of Lie groups simply isomorphic with no linear group. First we note an almost obvious lemma.

LEMMA 2. *Any non-Abelian hypercentral Lie group G has infinitesimal generators U, V, W satisfying $[U, V] = W$ and $[X, W] = 0$ for every X.*

PROOF. Since G is not Abelian, U_0 and V_0 exist satisfying $[U_0, V_0] = V_1 \neq 0$. Moreover either $[X, V_1] = 0$ for every X, or $[U_1, V_1] = V_2 \neq 0$ for some infinitesimal generator U_1. But since the Lie algebra of G is nilpotent, the latter alternative cannot recur indefinitely, and so we finally get elements U_q and V_q such that $[U_q, V_q] = V_{q+1}$, while $[X, V_{q+1}] = 0$ for every X.

LEMMA 3. *Any non-Abelian hypercentral Lie group G of matrices has a subgroup S_3 whose central lies in the central of G, topologically isomorphic in the large with G_3.*

PROOF. By Lemma 1, the U, V, and W of Lemma 2 generate a subgroup S_3 topologically isomorphic in the large with G_3, whose central therefore consists of the exp (λW). But since the exp (λW) commute with all the infinitesimal generators X of G, and G (being connected) is generated by these, the exp (λW) lie in the central of G.

THEOREM 3. *Any non-Abelian hypercentral Lie group G of matrices is locally topologically isomorphic with a Lie group G/N which is simply isomorphic with no linear group.*

PROOF. Refer to Lemma 3, and let N be the subgroup of the

exp (nW), n any integer. Then N is a discrete subgroup of the central of G, and so (see [1], p. 12) G/N is a Lie group locally topologically isomorphic with G.

But the homomorphism $G \rightarrow G/N$ carries S_3 into S_3/N, which is simply isomorphic with $G_3/N = G_3^*$. This and the corollary to Theorem 1 complete the proof.

E. Cartan [5] has shown that the universal covering group of the group of projective transformations of the line is topologically isomorphic in the large with no linear group.

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CHARACTERISTICS OF BIRATIONAL TRANSFORMS IN S_r

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1. *Introduction.* Consider a k-dimensional variety, V_k^n, of order n in an r-space, S_r. Let us project V_k^n from a general $(r-k-t-1)$-space of S_r upon a general $(k+t)$-space of S_r and denote the projection by $_t V_k^n$. We are supposing that $1 \leqq t \leqq k$. Then upon $_t V_k^n$ lies a double variety, D_{k-t}, of dimension $k-t$ and order b_t and upon D_{k-t} lies a pinch variety, W_{k-t-1}, of dimension $k-t-1$ and order j_{t+1}. Since the symbol W_{-1} is without meaning, we thus obtain $2k-1$ characteristics $b_1, b_2, \cdots, b_k$, $j_2, j_3, \cdots, j_k$. The symbol j_1 has a meaning which will be explained subsequently.

Now let a general $(r-k+q-2)$-space, $S_{r-k+q-2}$, $(1 \leqq q \leqq k)$, be given in S_r. Through this $S_{r-k+q-2}$ pass ∞^{k-q+1} primes of S_r and ∞^{k-q} of these are tangent to V_k^n. The points of contact form a $(k-q)$-dimensional variety, U_{k-q}. Denote its order by m_q. Thus