GENERALIZED JACOBI POLYNOMIALS

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1. *Introduction.* The differential equation

$$(\alpha x^2 + \beta x + \gamma)\frac{d^2y}{dx^2} - (x + a_1)\frac{dy}{dx} + \{n - n(n-1)\alpha\}y = 0,$$

where n is a positive integer, has polynomial solutions y_n of degree n. Some properties of these polynomials have been obtained by Brenke* and by Lawton.† The object of this paper is to derive fresh properties and in particular to study the zeros of these polynomials. Brenke proved that

$$h_n y_n = \frac{1}{\rho} D^n\{\rho P^n\},$$

where

$$P \equiv \alpha x^2 + \beta x + \gamma \equiv -\alpha(x-a)(b-x), \quad (a < b),$$

$$\rho \equiv -\frac{1}{\alpha}(x-a)^{A-1}(b-x)^{B-1},$$

$$A = \frac{a + a_1}{\alpha(b-a)}, \qquad B = \frac{b + a_1}{-\alpha(b-a)},$$

and h_n is the coefficient of x^n in the right-hand side. It has also been proved by him that if A and B are positive, the following recurrence formula holds good.

$$\text{(A)} \qquad y_n = (a_n + x)y_{n-1} - b_n y_{n-2},$$

where

$$b_n = \frac{c_{n-2}^2}{c_{n-1}^2} \quad \text{and} \quad \frac{1}{c_n^2} = \int_a^b \rho y_n^2\, dx,$$

$$a_n = -c_{n-1}^2 k_{n-1}, \quad \text{where} \quad k_n = \int_a^b x\rho y_n^2\, dx.$$

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* This Bulletin, vol. 36 (1930).

† This Bulletin, vol. 38 (1932).

The values of A and B have been assumed to be positive for the existence of the integral $\int_a^b \rho dx$.

2. *Values of y_n at a and b.*

$$
\begin{aligned}
h_n y_n &= \frac{1}{\rho} D^n(\rho P^n) = \frac{(-\alpha)^{n-1}}{\rho} D^n\left\{(x-a)^{n+A-1}(b-x)^{n+B-1}\right\} \\
&= (-\alpha)^n (x-a)^{1-A}(b-x)^{1-B} D^n\left\{(x-a)^{n+A-1}(b-x)^{n+B-1}\right\} \\
&= (-\alpha)^n (x-a)^{1-A}(b-x)^{1-B} \\
&\quad \cdot \left[\sum_{r=0}^{+n} (-1)^r C_{n,r} S_r (x-a)^{r+A-1}(b-x)^{n-r+B-1}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
S_r &= (n+A-1)(n+A-2)\cdots(A+r) \\
&\quad \times (n+B-1)(n+B-2)\cdots(n+B-r).
\end{aligned}
$$

Now $h_n = \alpha^n(2n+A+B-2)(2n+A+B-3)\cdots(n+A+B-1)$. Hence

$$
y_n(a) = (-1)^n \frac{(n+A-1)(n+A-2)\cdots A(b-a)^n}{(2n+A+B-2)(2n+A+B-3)\cdots(n+A+B-1)},
$$

$$
y_n(b) = \frac{(n+B-1)(n+B-2)\cdots B(b-a)^n}{(2n+A+B-2)(2n+A+B-3)\cdots(n+A+B-1)}.
$$

3. *Value of b_n in formula* (A).

$$
\begin{aligned}
\int_a^b \rho y_k^2 dx &= \frac{1}{h_k^2}\int_a^b D^k(\rho P^k)\left\{\frac{1}{\rho} D^k(\rho P^k)\right\} dx \\
&= \frac{(-1)^k}{h_k^2}\int_a^b \rho P^k D^k\left\{\frac{1}{\rho} D^k(\rho P^k)\right\} dx \\
&= \frac{(-1)^k k!}{h_k}\int_a^b \rho P^k dx \\
&= \frac{(-1)^k k!}{h_k}(-\alpha)^{k-1}\int_a^b (x-a)^{k+A-1}(b-x)^{k+B-1} dx \\
&= \frac{(-1)^k k!}{h_k}(-\alpha)^{k-1}\frac{\Gamma(k+A)\Gamma(k+B)}{\Gamma(2k+A+B)}(b-a)^{2k+A+B-1}.
\end{aligned}
$$

Hence

$$b_n = \frac{\int_a^b \rho y_{n-1}^2 dx}{\int_a^b \rho y_{n-2}^2 dx}$$

$$= \alpha \frac{h_{n-2}}{h_{n-1}} (n-1) \frac{(n+A-2)(n+B-2)}{(2n+A+B-3)(2n+A+B-4)} (b-a)^2$$

$$= (n-1) \frac{(n+A-2)(n+B-2)(n+A+B-3)}{(2n+A+B-3)(2n+A+B-4)^2(2n+A+B-5)} (b-a)^2.$$

4. *Relation among y_n', y_n, y_{n+1}.* From §2

$$h_n y_n = (-\alpha)^n \sum_{r=0}^{r=n} (-1)^r S_r C_{n,r} (x-a)^r (b-x)^{n-r}.$$

Hence

$$h_n y_n' = (-\alpha)^n \sum_0^n (-1)^r S_r C_{n,r} \{ r(x-a)^{r-1}(b-x)^{n-r} - (n-r)(x-a)^r(b-x)^{n-r-1} \}.$$

The coefficient

$$(x-a)^r(b-x)^{n-r-1}$$

on the right-hand side under summation is

$$\begin{aligned}
&(-1)^{r+1}C_{n,r+1}(r+1)S_{r+1} + (-1)^{r+1}(n-r)C_{n,r}S_r \\
&\quad = (-1)^{r+1}nC_{n-1,r}\{S_r + S_{r+1}\} \\
&\quad = (-1)^{r+1}nC_{n-1,r}\{(A+r) + (n+B-r-1)\}(n+A-1) \\
&\qquad\qquad \cdots (A+r+1) \times (n+B-1) \cdots (n+B-r) \\
&\quad = (-1)^{r+1}nC_{n-1,r}(n+A+B-1)(n+A-1) \\
&\qquad\qquad \cdots (A+r+1) \times (n+B-1) \cdots (n+B-r).
\end{aligned}$$

Therefore

$$\begin{aligned}
h_n y_n' &= -(-\alpha)^n n(n+A+B-1) \\
&\quad \times \sum_{r=0}^{n-1} \{ (-1)^r C_{n-1,r}(n+A-1) \cdots (A+r+1) \\
&\quad \times (n+B-1) \cdots (n+B-r) \times (x-a)^r(b-x)^{n-r-1} \}.
\end{aligned}$$

Now

$$\{(n+A)(b-x)-(n+B)(x-a)\}h_n y_n$$
$$= \{(n+A)(b-x)-(n+B)(x-a)\}(-\alpha)^n \cdot \sum_{r=0}^{n}(-1)^r S_r C_{n,r}(x-a)^r(b-x)^{n-r}.$$

The coefficient of $(x-a)^{r+1}(b-x)^{n-r}$ on the right-hand side is

$$(-\alpha)^n(-1)^{r+1}\{(n+B)C_{n,r}S_r+(n+A)C_{n,r+1}S_{r+1}\}$$
$$= (-\alpha)^n(-1)^{r+1}\{(n+B)(A+r)C_{n,r} + (n+A)(n+B-r-1)C_{n,r+1}\}$$
$$\times (n+A-1)\cdots(A+r+1)$$
$$\times (n+B-1)\cdots(n+B-r).$$

The expression within the braces is

$$(n+B)(n+A-\overline{n-r})C_{n,r}+(n+A)(n+B-\overline{r+1})C_{n,r+1}$$
$$= (n+A)(n+B)(C_{n,r}+C_{n,r+1})-nC_{n-1,r}(2n+A+B)$$
$$= (n+A)(n+B)C_{n+1,r+1}-nC_{n-1,r}(2n+A+B).$$

Hence

$$\{(n+A)(b-x)-(n+B)(x-a)\}h_n y_n$$
$$= (-\alpha)^n\sum_{0}^{n+1}(-1)^r C_{n+1,r}(n+A)(n+A-1)\cdots(A+r)$$
$$\times (n+B)\cdots(n+B-r+1)\times(x-a)^r(b-x)^{n-r+1}$$
$$+ (-\alpha)^n n(2n+A+B)$$
$$\cdot \sum_{0}^{n-1}(-1)^r C_{n-1,r}(n+A-1)\cdots(A+r+1)$$
$$\times (n+B-1)\cdots(n+B-r)(x-a)^{r+1}(b-x)^{n-r}$$
$$= \frac{h_{n+1}}{-\alpha}y_{n+1}+(-\alpha)^n n(2n+A+B)$$
$$\cdot \sum_{0}^{n-1}(-1)^r C_{n-1,r}(n+A-1)\cdots(A+r+1)$$
$$\times (n+B-1)\cdots(n+B-r)(x-a)^{r+1}(b-x)^{n-r}.$$

Also

$$
\begin{aligned}
(x - a)(b - x)h_n y_n' = &- n(n + A + B - 1)(- \alpha)^n \\
&\cdot \sum_0^{n-1} (- 1)^r C_{n-1,r}(n + A - 1) \cdots (A + r + 1) \\
&\times (n + B - 1) \cdots (n + B - r)(x - a)^{r+1}(b - x)^{n-r}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\{(n + A)(b - x) - (n + B)(x - a)\} h_n y_n \\
&\qquad = \frac{h_{n+1}}{- \alpha} y_{n+1} - \frac{2n + A + B}{n + A + B - 1} (x - a)(b - x)h_n y_n' .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(x - a)(b - x)y_n' = &- (2n + A + B - 1)y_{n+1} \\
&+ \frac{n + A + B - 1}{2n + A + B} \{(n + B)(x - a) - (n + A)(b - x)\} y_n.
\end{aligned}
$$

5. *Results Valid for A and B Negative.* If A and B be assumed to be negative, the results of §§2 and 4 obviously continue to hold good. The recurrence formula (A), being an identity and y_n''s being polynomials, will still be true and the form for b_n obtained in §3 will also hold good.

6. *Roots of $y_n = 0$.* Lawton and Fujiwara proved that if A and B be negative or zero, and p, q positive integers such that $0 < A + p \leqq 1$, $0 \leqq B + q \leqq 1$, the number of roots of $y_n = 0$ inside (a, b) is $n - p - q$, $(n \geqq p + q + 1)$. We shall derive the result by arguments based on methods of Sturm's theorem in the theory of equations and get fresh results by this consideration. The set of functions $y_n, y_{n-1}, \cdots, y_{p+q}$ is taken for this purpose, it being proved in the next section that $y_{p+q} = 0$ has no roots inside (a, b).

7. *No roots of $y_{p+q} = 0$ inside (a, b).* When x lies between a and b, $(x - a)$ and $(b - x)$ are both positive. The coefficient of $(x - a)^r(b - x)^{n-r}$ on the right-hand side of $h_n y_n$ in §2 is $(-\alpha)^n C_{n,r}(-1)^r S_r$. When $n = p + q$, this coefficient becomes

$$
\begin{aligned}
&(- \alpha)^{p+q} C_{p+q,r}(- 1)^r(p + q + A - 1) \cdots (A + r) \\
&\qquad\qquad \times (p + q + B - 1) \cdots \times (p + q + B - r).
\end{aligned}
$$

If $r = p - s$, where s is positive or zero,

$$(p+q+A-1)\cdots(p+A-1)\cdots(A+p-s)$$

has s negative factors and in $(p+q+B-1)\cdots(q+B+s)$ all the factors are positive. Hence in this case the sign of $(-1)^r S_r$ is that of $(-1)^p$. If, however, $r=p+t$, where t is positive, all the factors in $(p+q+A-1)\cdots(A+p+t)$ are positive and in $(p+q+B-1)\cdots(q+B-1)\cdots(q+B-t)$ there are t negative factors. Hence again the sign of $(-1)^r S_r$ is that of $(-1)^p$. It follows that $h_{p+q}y_{p+q}$ cannot vanish within (a, b) and has the same sign as that of $(-\alpha)^{p+q}(-1)^p$. Now

$$h_{p+q} = (+\alpha)^{p+q}(2p+2q+A+B-2) \cdots (p+q+A+B)(p+q+A+B-1).$$

Hence the sign of y_{p+q} is that of $(-1)^q(p+q+A+B-1)$, that is, the sign of y_{p+q} within (a, b) is that of $(-1)^q$ or $(-1)^{q-1}$ according as the sum of the fractional parts of $-A$ and $-B$ is <1 or >1. The case in which the sum $=1$ follows by continuity.*

8. *Changes in Sign.* In passing through a root θ of $y_n=0$ a change of sign will be lost or gained between y_n and y_{n-1} according as $y_n'(\theta)$ and $y_{n-1}(\theta)$ have like or unlike signs, that is, according as b_{n+1} is positive or negative (§4 and (A)). By formula (A) when $y_{k-1}=0$, y_k/y_{k-2} will have sign opposite to that of b_k. If b_k be positive, no change of sign will be lost or gained in passing through a root δ of $y_{k-1}=0$; if on the other hand b_k be negative, two changes of sign will be lost or gained according as $y_{k-1}'(\delta)$ and $y_{k-2}(\delta)$ are of the same or opposite sign. The relation of §4 enables us to deal with the different possibilities and an examination of the changes of sign lost enables us to determine the number of roots of $y_n=0$ within (a, b).

9. *Sign of b_n for Different Values of n.* For studying the zeros of y_n, $(n \geqq p+q)$, it is enough to consider the sign of b_n for $n \geqq p+q+2$. All the factors in b_n except $n+A+B-3$ are evidently positive. This factor is also positive for $n \geqq p+q+2$ if the sum of the fractional parts of $-A$ and $-B$ be less than unity, but if this sum exceeds unity, then b_n is negative for $n=p+q+2$, and positive for subsequent values of n.

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* This re-statement is made to guard against the possibility that we may have $h_{p+q}=0$ in the fraction $(-\alpha)^{p+q}(-1)^p/h_{p+q}$.

10. *Sum of Fractional Parts* <1. Thus if the sum of the fractional parts be <1, a change of sign will be lost in passing through a root of $y_n=0$ and no change of sign will be lost or gained for a root of any of the intermediate functions. Also from §2, it is evident that at b the functions are of the same sign and at a consecutive ones are of opposite signs. Hence the number of roots of $y_n=0$ between a and b will be $n-p-q$, $(n\geqq p+q)$.

11. *Sum of Fractional Parts* >1. If, on the other hand, the sum of the fractional parts >1, at $x=b$ the signs of y_{p+q} and y_{p+q+1} are opposite and there will be continuation of sign for the subsequent functions; whereas at a, y_{p+q} and y_{p+q+1} will be of the same sign and then there will be variation of sign at each stage subsequently. Hence the number of changes of sign lost will be $n-p-q-2$. The equation $y_{p+q+1}=0$ will evidently have only one root in (a, b). In passing through this root there will be two gains as b_{p+q+2} is negative in this case. In passing through a root of $y_n=0$, $(n\geqq p+q+2)$, there will be a loss of sign between y_n and y_{n-1} and there will be no loss or gain for a root of any $y_k=0$, $(p+q+2\leqq k\leqq n-1)$. Hence the number of roots of $y_n=0$ will exceed by 2 the number of changes of sign lost, that is, it will be $n-p-q$, $(n\geqq p+q)$. The result of Lawton is extended by our method to the case $n=p+q$.

12. *Application of Sturm's Method.* Sturm's method may be applied for the ranges $(-\infty, a)$ and (b, ∞) as well. In particular we shall show that the number of roots of the equation $y_{n''}=0$ and $y_{n'}=0$, where $n''>n'\geqq[-(A+B)+3]=n_0$, is the same in either of these ranges, $[x]$ denoting the integral part of x. For studying such a range it is clear that we cannot stop at y_{p+q}, but we have to take the whole set of functions down to y_0. Let us take the sets $y_0, y_1, \cdots, y_{n'}$ and $y_0, y_1, \cdots, y_{n''}$. In the first instance, the number of changes of sign in passing through $(-\infty, a)$ will be the same for the two sets. Also b_{n+1} is positive for $n\geqq n_0$. Hence in the second set no change is contributed for passing through a root of $y_{n'}=0, y_{n'+1}=0, \cdots, y_{n''-1}=0$, which come as intermediate functions. Again the nature of the change for a root of the last function in the two cases will be the same, for $y'_{n'}(\theta')/y_{n'-1}(\theta')$ and $y'_{n''}(\theta'')/y_{n''-1}(\theta'')$ are of the same sign $[y_n'(\theta)/y_{n'-1}(\theta)$ is negative in $(-\infty, a)$ for $n\geqq n_0]$. Now in the first set the number of losses is due to (1) the roots of some inter-

mediate functions and (2) the roots of $y_{n'}=0$. In the second set the contribution due to (1) persists. By the introduction of the additional functions in the second set, the change contributed by a root of $y_{n'}=0$ in the first set is transferred to a root of $y_{n''}=0$ in the second set. Also no change arises for a root of $y_{n'}=0$, $y_{n'+1}=0, \cdots, y_{n''-1}=0$. Since the total number of losses is the same in the two cases, the conclusion will be that the number of roots for $y_{n'}=0$ and $y_{n''}=0$ in the range $(-\infty, a)$ will be the same; similarly for the range (b, ∞). Also in (a, b) the numbers of roots of $y_{n'}=0$ and $y_{n''}=0$ are $n'-p-q$ and $n''-p-q$. Hence we conclude that the number of imaginary roots of $y_{n'}=0$ and $y_{n''}=0$ will be the same for $n''>n'\geqq n_0$.

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NOTE ON THE EXISTENCE OF AN nTH DERIVATIVE DEFINED BY MEANS OF A SINGLE LIMIT

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The nth derivative of a function $f(x)$ may be defined without the use of derivatives of lower order by means of the limit of a certain quotient. Conditions necessary and sufficient for the existence and continuity of $f^{(n)}(x)$ at a point $x=a$ and also for the mere existence of $f^{(n)}(a)$ have been recently given by Franklin.* The purpose of the present note is to state necessary and sufficient conditions of a somewhat more general form with proofs which use only Rolle's theorem and elementary properties of determinants.

Let $f_i(x)$ and $\phi_i(x)$, $(i=1, 2, \cdots, n+1)$, be functions possessing derivatives of the nth order, continuous in an interval I. Let $x_1, x_2, \cdots, x_{n+1}$ be points of I which close down in an arbitrary manner on a point a, in the sense that

$$(1) \qquad | x_j - a | < \epsilon_k, \qquad \lim_{k\to\infty} \epsilon_k = 0.$$

We shall use the notation

* This Bulletin, vol. 41 (1935), p. 573.