$$U(x, y) = \begin{cases} e, & x > 0, \\ 1, & x = 0, \\ e^{-1}, & x < 0. \end{cases}$$

The analogs of Theorems 3 and 5 in three or more dimensions require methods other than those developed in the present note, and are postponed for another occasion.

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ON SEQUENCES OF INDEFINITE INTEGRALS

BY M. K. GOWURIN

1. *Introduction.* The chief result concerning the subject of this paper is due to Lebesgue* and can be formulated as follows:

If $\{f_n(t)\}$ *is a sequence of functions defined and integrable in* $J=(0, 1)$ *and if for every measurable set* $e \subset J$

$$\lim_n \int_e f_n(t)dt = 0, \tag{1}$$

then the sequence of indefinite integrals

$$\int_e f_n(t)dt \tag{2}$$

is uniformly absolutely continuous.

G. Fichtenholz† has shown that the same conclusion remains true, if the equality (1) is satisfied for all open sets e.

S. Saks‡ considered the space $R=\{x\}$ of the characteristic

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* *Sur les intégrales singulières*, Annales de Toulouse, (3), vol. 1 (1909), p. 58; see also Hahn, *Über Folgen linearen Operationen*, Monatshefte für Mathematik und Physik, vol. 32 (1922), p. 45.

† *Theory of simple definite integrals depending on a parameter*, Petrograd, 1918, p. 98 (in Russian) or *Sur les suites convergentes des intégrales définies*, Bulletin de l'Académie des Sciences de Pologne, Sér. A, Décembre, 1923, pp. 115, 117.

‡ *On some functionals*, Transactions of this Society, vol. 35 (1933), pp. 549–556.

functions of measurable sets contained* in J, by introducing the metric:

$$\|x\| = \int_0^1 x(t)dt = mE_t(x(t) = 1), \quad d(x, y) = \int_0^1 | x(t) - y(t) | \, dt.$$

He proved the following generalization of Lebesgue's theorem:

If the equality (1) *written in the form*

$$\lim_n F_n(x) = \lim_n \int_0^1 f_n(t)x(t)dt = 0, \tag{1'}$$

is satisfied for all points x belonging to a certain set H of the second category in R, then the sequence of indefinite integrals (2) *is uniformly absolutely continuous, that is, the sequence of functionals*

$$F_n(x) = \int_0^1 f_n(t)x(t)dt \tag{2'}$$

is equally continuous in R, and $F_n(x) \to 0$ for all $x \in R$.

G. Fichtenholz has stated the following problem: *What is the general characteristic of the sets H in R, such that the equal continuity of the sequence* (2′) *follows from its convergence to zero*† *on H?* I shall study in particular the role of the category of H in this problem.‡

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* The space R is complete but not linear. The sum of two elements of R, $x+y$, exists if $x(t) \cdot y(t) = 0$, their difference $x-y$ exists if $x(t) \cdot y(t) = y(t)$. (Here as everywhere, all the equalities referring to the characteristic functions must be satisfied almost everywhere.) Further on we shall not prove the existence of sums and differences occurring in the text, since it is evident from the course of reasoning. The *zero* in the space R is the characteristic function of the vacuous set. Note also the equality $\|x+y\| = \|x\| + \|y\|$. Obviously, J can denote not only an interval, but every measurable set. We shall apply this statement in §3.

† As he showed, the convergence to zero of the sequence (2′) on all the R always follows from its convergence to zero on such a set H.

‡ [Added in proof.] As Saks has kindly pointed out to me, the totality of all the open sets considered by Fichtenholz is of the first category in R. Nevertheless, it seems to me that the example stated below is not without interest.

Without solving the problem stated above, I give in this paper an example which shows that in any case the category H is not the decisive factor in the question under consideration. Of course it is not at all difficult to construct an example, of a set H which is of the first category in R, but, being enlarged by all the existing linear combinations of its elements, forms a set of the second category in R. It is more natural, however, to consider the set H as being additive (that is, such that every existing linear combination of its elements belongs to it again). We shall prove the following statement.

THEOREM. *If the equality* $(1')$ *is satisfied for all points* x *belonging to some spherical surface in* R, $S^1(x_0, r)$, $(r<1)$, *which does not pass through the point* 0, *then the system* $(2')$ *is equally continuous in* R *and* $F_n(x)\rightarrow 0$ *everywhere in* R.

In general, the set $H=S(x_0, r)$ is not additive, but it is easy to see that the additive extension of $S(x_0, r)$ is nowhere dense in R.

2. *Proof of Theorem.* Let $x_0=0$. Then the spherical surface $S(0, r)$ consists of the points of norm r. Let us suppose that our assertion is wrong and that there exists a system $(2')$ converging to zero on $S(0, r)$, but not equally continuous. There exists an ϵ such that, for every $\delta>0$ and every integer N, R will contain an element x such that $\|x\|<\delta$ and $|F_n(x)|>\epsilon$ for a certain $n\geqq N$. Denote by H_m the set of points x of $S(0, r)$ such that $|F_n(x)|\leqq \epsilon/6$ for $n>m$. Evidently all H_m are closed and as $\sum_1^\infty H_m=S(0, r)$, at least one of them, for instance H_{m_0}, is of the second category in $S(0, r)$. Being closed, H_{m_0} contains in $S(0, r)$ a sphere $T(0, r;\ \xi_0, \rho)$ or, more briefly, T (T consists of such points x that $\|x\|=r$ and $d(x, \xi_0)\leqq\rho$).

According to the assumption there exists in R an element x_1 such that

$$\|x_1\| = \alpha_1 < \frac{r}{3}, \qquad \alpha_1 < \frac{1-r}{3}, \qquad \alpha_1 < \frac{\rho}{4},$$

and for some $\nu>m_0$, $|F_\nu(x_1)|>\epsilon$. For example, let $F_\nu(x_1)>\epsilon$.

Consider an arbitrary element $\bar{x}$ of the spherical surface $S(0, \alpha_1)$. Construct such an element z that $z+x_1$ and $z+\bar{x}$ exist and are contained in T. The method of construction depends upon the quantity $\beta=\|(x_1+\bar{x}-x_1\bar{x})\xi_0\|$, which can be equal to, greater, or less than α_1.

In the first case $z=\xi$, where $\xi=\xi_0-(x_1+\bar{x}-x_1\bar{x})\xi_0$.

In the second case we put $z=\xi+\eta$, where η is the characteristic function of a set of measure $\beta-\alpha_1$, lying in $E_t(\xi_0(t)+x_1(t)+\bar{x}(t)=0)$. It follows from $\beta-\alpha_1\leqq\alpha_1$ and from

$$mE_t(\xi_0(t)+x_1(t)+\bar{x}(t)=0)\geqq 1-(\|\xi_0\|+\|x_1\|+\|\bar{x}\|)$$
$$=1-r-2\alpha_1\geqq 1-r-2\frac{1-r}{3}=\frac{1-r}{3}>\alpha_1,$$

that such a set exists.

In the third case $z=\xi-\chi$, where χ is the characteristic function of a set of measure $\alpha_1-\beta$ lying in $E_t(\xi(t)=1)$. It follows from

$$mE_t(\xi(t)=1)=\|\xi\|\geqq r-2\alpha_1>\alpha_1,$$

that such a set exists.

In each of these cases $\|z\|=r-\alpha_1$ and $d(z,\xi_0)\leqq 3\alpha_1$. Evidently z satisfies the conditions indicated above. Then

$$F_\nu(z+x_1)=F_\nu(z)+F_\nu(x_1),\qquad F_\nu(z+\bar{x})=F_\nu(z)+F_\nu(\bar{x});$$
$$F_\nu(\bar{x})=F_\nu(z+\bar{x})-F_\nu(z)=F_\nu(z+\bar{x})-F_\nu(z+x_1)+F_\nu(x_1).$$

As $z+x\in T$ and $z+\bar{x}\in T$, the first two members of the right part of the last equality are numerically $<\epsilon/6$, whence $F_\nu(\bar{x})>4\epsilon/6$.

Note that $\bar{x}$ is an arbitrary element of $S(0,\alpha_1)$. Let α_2 be the least non-negative residue of r mod α_1, that is, $r=k_1\alpha_1+\alpha_2$, (k_1 a positive integer, $0\leqq\alpha_2<\alpha_1$). Then ξ_0 can be represented in the form

$$\xi_0=u_1+u_2+\cdots+u_{k_1}+x_2,$$

where $u_i\cdot u_j=0$, $(i\neq j)$, $u_ix_2=0$, $\|u_i\|=\alpha_1$, and $\|x_2\|=\alpha_2$.

We have

$$F_\nu(x_2)=F_\nu(\xi_0)-F_\nu(u_1)-\cdots-F_\nu(u_{k_1}).$$

Since $|F_\nu(\xi_0)|<\epsilon/6$ and $F_\nu(u_i)\geqq 4\epsilon/6$, we have

$$F_\nu(x_2)<\frac{\epsilon}{6}-k_1\frac{4\epsilon}{6}\leqq-\frac{3\epsilon}{6}.$$

Repeating for x_2 and $S(0, \alpha_2)$ the reasoning given for x_1 and $S(0, \alpha_1)$, we find for $x \in S(0, \alpha_2)$, $F_\nu(x) < -\epsilon/6$.

Dividing α_1 by α_2, we obtain

$$\alpha_1 = k_2\alpha_2 + \alpha_3, \qquad (k_2 \text{ a non-negative integer, } 0 \leqq \alpha_3 < \alpha_2).$$

If $x \in S(0, \alpha_3)$, then adding to it k_2 suitably selected elements $v_1, v_2, \cdots, v_{k_2}$ belonging to $S(0, \alpha_2)$, $(v_i \cdot v_j = 0$, when $i \neq j$, and $v_i \cdot x = 0)$, we obtain an element $y \in S(0, \alpha_1)$. Then

$$F_\nu(x) = F_\nu(y) - F_\nu(v_1) - \cdots - F_\nu(v_{k_2}),$$

and

$$F_\nu(x) > \frac{4\epsilon}{6} + k_2 \frac{\epsilon}{6} \geqq \frac{5\epsilon}{6}.$$

Divide α_2 by α_3, and so on. If r and α_1 are incommensurable, we shall obtain a sequence of spheres $S(0, \alpha_p)$ converging to zero, where

on $S(0, \alpha_1)$, $F_\nu(x) > 4\epsilon/6$,
on $S(0, \alpha_2)$, $F_\nu(x) < -\epsilon/6$,
on $S(0, \alpha_3)$, $F_\nu(x) > 5\epsilon/6$,
on $S(0, \alpha_4)$, $F_\nu(x) < -6\epsilon/6$,
on $S(0, \alpha_5)$, $F_\nu(x) > 11\epsilon/6$,

and so on. Such a result contradicts the continuity of $F_\nu(x)$. If r and α_1 are commensurable, a certain α_μ will be a divisor of r. Then

$$\xi_0 = w_1 + w_2 + \cdots + w_\lambda,$$

where $w_i \cdot w_j = 0$, $(i \neq j)$, and $w_i \in S(0, \alpha_\mu)$, $(i = 1, 2, \cdots, \lambda)$. The numbers $F_\nu(w_i)$ have the same sign for every i and $|F_\nu(w_i)| > \epsilon/6$. Consequently, $|F_\nu(\xi_0)| > \epsilon/6$, which is impossible, since $\nu > m_0$ and $\xi_0 \in H_{m_0}$. Thus the system $(2')$ is really equally continuous.

Thus every sequence $(2')$, which satisfies condition $(1')$ on $S(0, r)$ is equally continuous. Hence it follows that such a sequence converges to zero everywhere* on R, according to the general assertion.

* The reader sees that as far as it concerns the spherical surface $S(0, r)$ with the center at the point $x = 0$, the proof will not have to be changed considerably if we suppose that H, although not coinciding with $S(0, r)$, is of the second category in $S(0, r)$.

3. *Application of Functionals.* Now let $S(x_0, r)$ be any spherical surface which does not pass through 0 and suppose that, for $x \in S(x_0, r)$, $\lim_n F_n(x) = 0$. Note that every spherical surface in R has two centers and divides the whole space into two spheres, that is, that $S(x_0, r) = S(1-x_0, 1-r)$. Suppose that x_0 is that one of the two centers whose norm is greater than the radius.

For further reasoning it is necessary to show that $F_n(x_0) \to 0$. For that purpose consider the set of points $x \in S(x_0, r)$ for which $x \cdot x_0 = x$ and denote it by S'. If we introduce the set $J' = E_t(x_0(t) = 1)$, then $S' = S'(0, \|x\| - r)$ is a spherical surface with the center at 0 and with radius $\|x_0\| - r$ in the space R' of characteristic functions of measurable subsets of J'. Since reasoning in §2 remains valid when J' is substituted for J, we have $F_n(x) \to 0$ everywhere in R' and in particular at the point x_0.

Any element $x \in R$ can be represented in the form $x = x_0 - u(x) + v(x)$, where $u(x) = x_0(1-x)$ and $v(x) = x(1-x_0)$. Let us define a one-to-one and bicontinuous transformation of R into itself according to the formulas

$$y = \Psi(x) = u(x) + v(x) = x_0(1 - x) + x(1 - x_0),$$

$$x = \Psi^{-1}(y) = x_0 - yx_0 + y(1 - x_0).$$

In this way $S(x_0, r)$ is transformed into $S(0, r)$.

Further define in the transformed space the sequence of functionals $\{\Phi_n(y)\}$ on putting

$$\text{(A)} \qquad \Phi_n(y) = F_n(x) - \frac{F_n(x_0)}{\|x_0\|} \int_0^1 x_0(t) x(t) dt.$$

If we remember the definition of $F_n(x)$ and express x in terms of y, we easily find that the functionals $\Phi_n(y)$ have the form required,

$$\Phi_n(y) = \int_0^1 g_n(t) y(t) dt.$$

Since $F_n(x_0) \to 0$ and $F_n(x) \to 0$ on $S(x_0, r)$, we conclude from the equality (A) that $\Phi_n(y)$ converges to zero on $S(0, r)$. Applying the result of §2 to the sequence $\{\Phi_n(y)\}$, we see that

$\Phi_n(y) \to 0$ everywhere in R. Since the transformation is one-to-one, it follows from the formula (A) that $F_n(x) \to 0$ everywhere in R.

Note that the statement made in §1 is not true for a spherical surface which passes through zero, $S(x_0, \|x_0\|)$. This we can prove on putting

$$F_n(x) = n \int_0^1 [x_0(t) - (1 - x_0(t))] x(t) dt.$$

Let $x \in S(x_0, \|x_0\|)$; $x = x_0 - u + v$ and

$$\begin{aligned} \|x_0\| = d(x, x_0) &= \int_0^1 | x(t) - x_0(t) | dt \\ &= \int_0^1 | -u(t) + v(t) | dt \\ &= \int_0^1 [u(t) + v(t)] dt = \|u\| + \|v\|, \end{aligned}$$

that is, $\|x_0 - u\| = \|v\|$. According to definition, $x_0 - u = x_0 \cdot x$ and $v = x(1 - x_0)$. Consequently for such an x, $F_n(x) = 0$, for $n = 1, 2, \cdots$. At the same time the sequence $\{F_n(x)\}$ is by no means equally continuous in R, as $|F_n(x)| \to \infty$ for every x non-$\in S(x_0, \|x_0\|)$.

4. *Remark.* In the paper of Saks mentioned above there are some vague points in the proof of Lemma 4. This refers to the way he motivates the inclusion (on page 553)

$$H \subset H_1 + H_2 + \cdots.$$

The lemma itself is not true, at least for the space R, as is shown by the example of the following sequence of transformations:

$$\xi_n(x, t) = nx(t).$$

Moreover, the assertions of Theorems 3 and 4 are not true in so far as they refer to R, as is seen from the same example.

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