Von Mises has attempted to take an intermediate position between these two points of view. His own point of view is not clearly defined, however, and most criticism has supposed that he adopted the first. His principal justification has been that no contradiction will be derived, using his axioms. Now it can be shown that the ordinary probability calculus can be developed fully using his axioms, and that in such a development no contradiction will ever be obtained—the axioms lead to a consistent set of rules of procedure. But absence of contradiction on such a level cannot be the main justification of a mathematical theory to any mathematician who believes his science is more than a chess-like game: surely a set of rules of procedure should have an acceptable base. What is desired is a mathematical theory which runs parallel to the physical facts, when properly idealized, but which has its own independent justification.

This edition of *Wahrscheinlichkeit Statistik und Wahrheit* contains a considerably enlarged critique of various theories of probability which will be of lasting value to all students of the subject.

J. L. Doob

Wahrscheinlichkeitsrechnung und allgemeine Integrationstheorie. By E. Tornier. Leipzig and Berlin, Teubner, 1936. 6+158 pp.

In the last few years, the theory of probability has been more and more influenced by the modern theories of measure. Professor Tornier gives a striking proof of this in devoting 100 of the 158 pages of his *Wahrscheinlichkeitsrechnung* to an interesting and fairly complete development of (Jordan) content and (Lebesgue) measure theories—treated from an abstract standpoint. The reader is warned in the introduction not to be deterred by this heavy array of pure mathematics: "so much mathematics is needed precisely in order to avoid reducing living basic intuitions into lifeless formalism, as results, for example, from an identification of probability with Lebesgue measure—inspired by the analogy in the rules of calculation." As we shall see, the author rejects Lebesgue measure in favor of Jordan measure, thus avoiding lifeless formalism.

Consider the theory of probability as applied to the analysis of the repeated casting of a single die, marked in the usual way. Any sequence $(n_1, n_2, \cdots)$ is logically possible, where n_j is one of the integers 1, 2, 3, 4, 5, 6. Tornier assigns probabilities to certain classes of these sequences. Thus to the class of all sequences for which $n_1=4$ (representing the possibility of casting a 4 the first time), is assigned the probability 1/6. More generally, if $a_1, \cdots, a_\nu$ is any finite set of integers between 1 and 6, the class of all sequences for which $n_j=a_j, j=1, \cdots, \nu$, is given probability $1/6^\nu$. These sets of sequences are called basic sets, and assigning these probabilities to the basic sets and prescribing the usual additive property of probability determines a probability measure—a set function defined on certain sets of sequences. This probability measure can be taken as (Jordan) content or (Lebesgue) measure, depending on the extent of the field of sets on which probability is defined. Now the author uses in a fundamental way special classes of sequences $(n_1, n_2, \cdots)$ having an intimate connection with the field of Jordan measurable sets determined by the basic sets, and this connection cannot be extended to the more general field of

Lebesgue measurable sets determined by the basic sets. It is through these special sequences that the probability measure is related to the usual frequency interpretation of probability. It is therefore impossible for him to assign probabilities to all Lebesgue measurable sets and retain the frequency interpretation, and he concludes that it is impossible to combine a frequency concept with the use of all measurable sets. That the difficulty is not in the problem but in the attack is shown by many papers in the literature which utilize the field of Lebesgue measurable sets on this very space of sequences, and do not abandon the frequency interpretation.* Tornier does not refer to such papers, but claims that the identification of probability with Lebesgue measure has "deprived probability theory of its independent existence, and made it into an empty mathematical formalism without content and without any connection with the external world, whose calculated results even lack any conceivable possibility of experimental verification." Such strong language was unusual in the mathematical works of a less heroic era. Even now it should be followed by some supporting evidence. It should be noted parenthetically that an acceptance of the formal identity of the rules of probability with the rules of measure theory, as applied to suitable spaces, would by no means imply the identity of probability and Lebesgue measure.

A striking example of the difference in results in using content and measure is one mentioned by Tornier: in the experiment discussed above, consider the event consisting of casting only 3's with the die, after some stage. The corresponding class of sequences $(n_1, n_2, \cdots)$ is not in the Jordan field over the basic sets, so the Tornier method does not give the event a determinate probability p, and Tornier shows that any number between 0 and 1 (inclusive) can consistently be assigned to it. There is no indication as to how p is to be chosen, or whether its choice is determined at all by the actual experiment. The other approach assigns to this event the probability 0. The use of Jordan measure also causes certain changes in well known theorems. For instance if $\mathbf{x}$ is a chance variable, the probability that $\mathbf{x}<c$ may not be defined for all values of c: a denumerable exceptional set is possible. This fact changes such statements as that of Markoff's Lemma in an obvious way.

The author's purpose was to show how his approach leads to a consistent theory of probability, and he has done this in a clearly written book.

J. L. Doob

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* Cf. for example A. Khintchine, Zeitschrift für angewandte Mathematik und Mechanik, vol. 13 (1933), pp. 101–103; E. Hopf, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 13 (1934), pp. 51–102; J. L. Doob, Transactions of this Society, vol. 36 (1934), pp. 759–775; S. Ulam and Z. Lomnicki, Fundamenta Mathematicae, vol. 23 (1934), pp. 237–278. Numerous earlier papers adopted the same point of view without using the specific terminology of probability.