LINEAR OPERATIONS ON FUNCTIONS OF BOUNDED VARIATION

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The object of this note is to give a form for the most general linear continuous operation on the space of functions of bounded variation on a finite interval, say $0 \leqq x \leqq 1$, the norm of the space being the total variation.

This form is obtained by setting up an equivalent space. For this purpose let $\mathfrak{J}$ be the class of elements I consisting of any finite number of non-overlapping intervals $i_1, \cdots, i_n$ of the interval (0, 1). If (x_p, y_p) are endpoints of i_p, define the function of interval sets $\beta(I) = \sum_{p=1}^n [\alpha(y_p) - \alpha(x_p)]$ corresponding to the function $\alpha(x)$ of bounded variation. Then $\beta(I)$ is a bounded function on $\mathfrak{J}$. Define $\|\beta\|$ in the usual way as the least upper bound of $|\beta(I)|$ for I on $\mathfrak{J}$. Then the space $\mathfrak{B}$ of additive set functions β thus normed is equivalent to the space $\mathfrak{A}$ of functions $\alpha(x)$ of bounded variation with $\|\alpha\| = V\alpha = \int_0^1 |d\alpha|$,* for obviously $\|\beta\| \leqq \|\alpha\| \leqq 2\|\beta\|$. Further, if α_1 corresponds to β_1 and α_2 to β_2, then $\beta_1 + \beta_2$ corresponds to $\alpha_1 + \alpha_2$ and $c\beta$ to $c\alpha$, and conversely.

It is now an easy matter to determine the most general linear functional operation on the space $\mathfrak{B}$. Following the lines of reasoning of my paper *On bounded linear functional operations*,† one finds that for any linear continuous operation L on the space $\mathfrak{B}$ there exists an additive function γ of sets E of elements I, such that $L(\beta) = \int \beta d\gamma$, the integral being of the L or S type as defined in the paper quoted, and extended over the class of all subsets of elements of $\mathfrak{J}$. Because of the relationship between the functions β and α this gives the most general linear operation in the space $\mathfrak{A}$.

It might be noted that a similar reasoning applies to the set of interval functions $\alpha(i)$ where $\sum_{p=1}^n \alpha(i_p) = \beta(I)$ is a bounded function on $\mathfrak{J}$; or, more generally, that a similar result holds in the space of bounded functions on a general range, with norm the least upper bound of the absolute value of the function on the range.

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* Note that in the space $\mathfrak{A}$ two functions for which $V(\alpha_1 - \alpha_2) = \int |d(\alpha_1 - \alpha_2)| = 0$ are regarded as equivalent. To obtain uniqueness, the condition $\alpha(0) = 0$ can be added. If we wish that $\|\alpha\| = 0$ imply $\alpha = 0$ for all x, we may choose $\|\alpha\| = |\alpha(0)| + V\alpha$. The space $\mathfrak{B}_1$ corresponding is defined by $\beta_1(I) = \alpha(0) + \sum_{p=1}^n [\alpha(y_p) - \alpha(x_p)] = \alpha(0) + \beta(I)$ and $\|\beta_1(I)\| = |\alpha(0)| + \|\beta(I)\|$. Reasoning similar to the above can be carried through in this case also.

† Transactions of this Society, vol. 36 (1934), pp. 868–875.