NOTE ON CONVEX REGIONS ON THE SPHERE*

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By a convex region on the sphere we mean a region such that any great circle arc of length less than 180°, whose end points lie in the region, lies entirely in the region. Let G denote any convex region, G_0 the diametrically opposite region, and G_1 the set obtained from the whole sphere by excluding G and G_0 together with their boundaries. Let $\rho = \arctan(3^{1/2}/2)$; then $40°53' < \rho < 40°54'$. We shall prove the following theorem.†

There is on the sphere a circle-interior of radius ρ (measured on the sphere), which lies entirely in G or entirely in G_1. The number ρ cannot be replaced by any larger number.

Let r be the least upper bound of the radii of circle-interiors lying in G. Then for every integer $n > 1$ there is a circle-interior of radius $r(1-1/n)$ lying in G. A limit point of their centers is the center of a circle-interior C of radius r lying in G. We may suppose that G is neither the whole sphere nor a hemisphere, so that $r < 90°$.

No closed semicircumference forming part of the boundary of C can be free of boundary points of G. For if it were, it would be at a distance $d > 0$ from the boundary of G. Thus G would contain not only C but all points within a distance d from one-half of its circumference. Hence G would contain a circle-interior of radius greater than r.

If P is any boundary point of G on the circumference of C, then G lies entirely on one side of the great circle tangent to C at P. For if there were a point P' of G on the opposite side of this great circle from C, we could join P' to P by a great circle arc of less than 180°, which when extended through P would cut C. Taking P'' as a point sufficiently near to P on $P'P$ extended, we should have $P'P'' < 180°$, P on $P'P''$, P' and P'' in G, therefore P in G, contrary to hypothesis.

We shall show that there is an arc PQ forming not more than one-half nor less than one-third of the circumference of C, and whose end points P and Q are boundary points of G. Let P_1 be any boundary point of G on the circumference of C, and let P_0 be the opposite point of the circumference. If P_0 is a boundary point of G, then we may take either arc P_1P_0 as PQ. Otherwise let P_2 and P_3 be the boundary

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† This theorem was suggested to me by Hans Lewy, who makes use of it in a current paper.

points of G on the circumference of C nearest to P_0 on either side. Then the arcs P_1P_2 and P_1P_3 are less than a semicircumference by construction, and P_2P_3 cannot be greater than a semicircumference, since its interior is free of boundary points of G. The longest of the three arcs is at least one-third of the circumference, and hence can be taken as PQ.

Drawing the tangent great circles to C at P and Q, we have G included within a lune, whose angle can be easily calculated (in terms of r and the length of PQ). Let 2θ be the angle of the lune, and 2α the angle subtended by PQ at the center of C. Then $60° \leqq \alpha \leqq 90°$. Join the nearer point of intersection of the tangents at P and Q to the center of C; also join P to the center of C. A right triangle is formed with angles α and θ, and side r opposite the latter angle. Applying a formula true for any right spherical triangle, we have

$$\cos \theta = \cos r \sin \alpha,$$

hence $\cos \theta \geqq (3^{1/2}/2) \cos r$. Let $r' = 90° - \theta$. Then in either of the supplementary lunes there is a circle-interior of radius r', and

$$\sin r' \geqq (3^{1/2}/2) \cos r.$$

For values of r and r' satisfying this inequality, the smallest value of $\max(r, r')$ will occur for $\sin r' = (3^{1/2}/2) \cos r$, and $r' = r$. The common value of r and r' is then $\rho = \arctan (3^{1/2}/2)$. Hence in every case $\max(r, r') \geqq \rho$, so that there is always a circle-interior of radius ρ in G or G_1.

We must show that ρ cannot be replaced by any larger number. For this purpose we find a particular region G so that there is no circle-interior of radius greater than ρ in G or G_1. Let G be the interior of an equilateral triangle circumscribed about a circle of radius ρ. If the angles of the triangle are 2θ, we have (putting $r = \rho$ and $\alpha = 60°$) $\cos \theta = (3^{1/2}/2) \cos \rho = \sin \rho$, or $\theta = 90° - \rho$. The angles of the triangle being more than 90°, the sides are also more than 90°. The exterior angles of the triangle are 2ρ. In G_1 there are then six circles of radius ρ, touching one side of G and one side of G_0. There are no larger circles in G_1. For every maximum circle, since it cannot touch G and G_0 at three points, must touch them at two opposite points of the circle. This can happen only for the six circles mentioned and for the six circles which pass through a vertex of one triangle and are tangent at the mid-point of a side of the other. These latter circles are seen, however, not to furnish even a relative maximum.

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