ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

90. R. P. Agnew: *Cores of complex sequences and of their transforms.*

It is shown that a regular transformation $\sigma_n=\sum_{k=1}^{n}a_{nk}s_k$, determined by a matrix a_{nk} of complex constants, transforms each definitely divergent sequence s_n (K. Knopp, Mathematische Zeitschrift, vol. 31, pp. 97–127) into a definitely divergent sequence σ_n if and only if there is an index K such that a_{nk} is real and non-negative for all $k\geqq K$. This condition is also necessary and sufficient to ensure that the core ("Kern" of Knopp, loc. cit.) of each sequence s_n contain the core of its transform σ_n. Some related results are obtained, including a characterization different from but equivalent to Knopp's definition of definite divergence. Finally a theorem of Steinhaus type (Prace Matematyczno-fizyczne, vol. 22, pp. 121–134) is given with comments relevant to cores. (Received January 17, 1938.)

91. A. A. Albert: *Non-cyclic algebras with pure maximal sub-fields.*

The author modifies the algebras of his paper in the Transactions of this Society, vol. 35 (1933), pp. 112–121, and thereby proves the existence of normal division algebras of degree four over a non-modular field K which are non-cyclic yet have maximal sub-fields $K(j)$, $j^4=g$ in K. This proves the falsity of a recent conjecture on normal division algebras (see introduction of the paper in the Transactions of this Society, vol. 39 (1936), pp. 183–188). (Received January 22, 1938.)

92. E. F. Beckenbach and M. Reade: *Generalizations to space of the Cauchy and Morera theorems.*

The theorems of Cauchy and Morera are generalized for three-space. A typical result is the following: If $x_j(u,v)$, $(j=1,2,3)$, are harmonic functions in a domain D, then a necessary and sufficient condition that these functions be the coordinate functions of a minimal surface given in isothermic representation is that for each circle C, lying in and enclosing only points of D, $\sum_{j=1}^{3}[\int_C x_j(u,v)\cdot(du+idv)]^2=0$. (Received January 26, 1938.)

93. Garrett Birkhoff: *Dependent probabilities and spaces (L).*

The paper bases the theory of dependent probabilities on the theory of partially ordered function spaces. This formulation makes it possible to include all known cases in a single set of definitions (cf. also abstracts 43-9-320-321). A new theorem, which specializes in the *deterministic* case to von Neumann's mean ergodic theorem, is proved with added generality in the *stochastic* case. (Received January 28, 1938.)

94. Garrett Birkhoff: *Partially ordered linear spaces.*

A "K-space" is defined (Kantorovitch) as a linear space over the real field, which is a lattice, in which $x \to x+a$ and $x \to \lambda x$, $(\lambda > 0)$, preserve order, and $x \to -x$ inverts order. The "normal subspaces" corresponding to homomorphisms which preserve inclusion are characterized abstractly; they form a *distributive* lattice, and the components of direct decompositions of L form the Boolean algebra of complemented elements in this lattice. Any "bounded" additive functional λ (that is, one bounded on bounded subsets of L) is the sum of its positive and negative parts; hence, the bounded additive functionals form a "conjugate" K-space. There is just one K-space of each finite dimension n. Define a K-space as "continuous" when every bounded monotone sequence has a limit, $\lambda(x)$ as "continuous" if $x_n \uparrow x$ implies $\lambda(x_n) \to \lambda(x)$. Under these added hypotheses, L is the direct sum of three components—on which λ is negative, zero, and positive, respectively. If λ is positive (that is, if $f > 0$ implies $\lambda(f) > 0$), then L is a linear metric space relative to $\lambda(|f-g|)$, and a Banach space of type (L) if and only if L is "continuous." (Received January 28, 1938.)

95. R. P. Boas (National Research Fellow) and Salomon Bochner: *Closure theorems for translations.*

Let D denote the space of complex-valued functions $g(t)$ defined on $(-\infty, \infty)$, continuous except for discontinuities of the first kind, and such that $g(\infty)$ and $g(-\infty)$ exist; let C denote the subspace of continuous functions $f(t)$ such that $f(\infty) = f(-\infty) = 0$. Every $g(t)$ of D has a generalized Fourier transform $G(t)$ of order 2 (as defined by S. Bochner, *Vorlesungen über Fouriersche Integrale*, chap. 6). It is possible to approximate uniformly on $(-\infty, \infty)$ to any $f(t)$ of C by finite linear combinations of translations, $g(t+u)$, of $g(t)$ of D if and only if $G(t)$ is linear on no interval. The proof depends on theorems from operation theory, Kaltenborn's determination of the general linear functional on D (this Bulletin, vol. 40, p. 702), and theorems used by Wiener in discussing the closure of translations of functions of L. Similar results are obtained when one admits functions $g(t)$ which are $O(|t|^n)$, $(|t| \to \infty;\ n = 1, 2, \cdots)$. (Received January 25, 1938.)

96. J. W. Calkin: *Abstract self-adjoint boundary conditions.*

Let L be an arbitrary differential operator applicable to functions defined over an open set E in euclidean n-space and coincident with its formal adjoint. One may then consider the problem of determining, when possible, a lucid definition of the boundary conditions defining those domains in the Hilbert space $\mathfrak{L}_2(E)$ on which L induces self-adjoint transformations. Aside from the matter of closure, this problem involves in large part questions of an algebraic or quasi-algebraic nature. It is a primary function of the present work to provide an abstract algebraic framework within which such questions can be clearly answered, under certain reasonable restrictions on L. The author has found that the results apply to various partial differential operators, among them operators of the form $-\partial(p\partial/\partial x)/\partial x - \partial(p\partial/\partial y)/\partial y + q$ (cf. abstract 43-3-114) and their iterates, as well as to ordinary differential operators. (Results for ordinary differential operators have been obtained directly by M. H. Stone and I. Halperin.) The theory also has other implications; in particular, it contains in essence an independent development of von Neumann's theory of Cayley transforms, motivated quite differently from the original. A brief account of some of the more fundamental results appears in the Proceedings of the National Academy of Sciences, vol. 24 (1938), no. 1. (Received January 28, 1938.)

97. Richard Courant: *The existence of minimal surfaces of least area bounded by prescribed manifolds.*

By means of a general theorem concerning potential functions, the existence proof for minimal surfaces of least area is given if the boundary consists of prescribed Jordan arcs and point sets free to lie on prescribed continuous manifolds of more than one dimension. The topological structure of the minimal surface can be prescribed under certain inequality conditions. (Received January 18, 1938.)

98. H. S. M. Coxeter: *A plane projection of the polytope* 2_{21} *whose* 27 *vertices represent the lines on the general cubic surface.*

Among the 27 lines on a cubic surface, three (such as c_{12}, c_{56}, c_{16}), of which two intersect while the third is skew to both, uniquely determine a fourth (c_{25}) which intersects the third but is skew to the first two. This suggests a representation of the 27 lines by 27 points of a plane, the lines of such an "XX configuration" being represented by the vertices of a parallelogram. Pairs of the points are joined or not joined according as they represent skew or intersecting lines. One of the most symmetrical arrangements is obtained by beginning with a regular enneagon, whose vertices represent a cycle of nine lines such as a_1, c_{35}, c_{56}, b_2, c_{16}, b_5, c_{12}, c_{24}, a_6. The rest of the points follow immediately by completing parallelograms; they form two smaller enneagons, concentric with the first. The whole figure can be regarded as an orthogonal projection of the vertices and edges of Gosset's six-dimensional semi-regular polytope 2_{21}, whose relation to the cubic surface was noticed by Schoute and explained by Du Val (Proceedings of the London Mathematical Society, (2), vol. 35 (1932), pp. 23–74). The parallelograms arise as projections of squares. (Received June 29, 1937.)

99. J. J. De Cicco: *Fields whose geodesic series can be represented by the turbines of a flat field.*

This paper is a continuation of the paper, *Fields whose geodesic series have circles for their point- or line-unions* (abstract 43-11-377), by the author. A transformation between two fields is said to be *equi-deviate* if the distance between the centers of the tangent turbines at a common element of any two field series is preserved. In this paper the following two results are proved: 1. In order that a field be mapped into a flat field in such a way that the geodesic series of the field correspond to the turbines (the geodesic series) of the flat field, it is necessary and sufficient that the curvature G of the field be the same for all elements of any equi-parallel series of the field. 2. In order that a field be mapped by an equi-deviate transformation into a flat field in such a way that the geodesic series of the field correspond to the turbines of the flat field, it is necessary and sufficient that the curvature G of the field be constant and less than one. The analogous problem in three-space was discussed by Kasner in the paper, *The generalized Beltrami problem concerning geodesic representation*, Transactions of this Society, vol. 4 (1903), no. 2. (Received January 10, 1938.)

100. Arnold Dresden: *An approach to existence theorems in the calculus of variations.* Preliminary report.

By using the fact that a lower semicontinuous function, defined in a metric space, has extreme values on any self-compact set in that space, the author intends to establish existence theorems for a variety of functionals occurring in the calculus of variations. The problem falls into two parts: (1) the determination of conditions on the integrand function which insure lower semicontinuity of the functional on the space of absolutely continuous functions; for this part the work of Tonelli, McShane, and others is drawn upon; (2) the setting up of conditions under which the classes of absolutely continuous functions occurring in the various functionals are self-compact. The first part needs to be done only once for all functionals in which the same integral occurs. The second part requires a study of the class of absolutely continuous functions and of successive subclasses. This preliminary communication discusses the general mode of approach and announces results, many of which have been obtained before, some of which are believed to be new. (Received January 27, 1938.)

101. Melvin Dresher and Oystein Ore: *On the theory of multigroups.*

This paper contains a theory of multigroups (hypergroups), that is, group-like systems in which the multiplication is not one valued. The results of Marty and Wall are extended and a general discussion of the structure properties is given. A certain class of multigroups whose composition series consist of ordinary groups is of particular interest. (Received January 25, 1938.)

102. J. E. Eaton: *A formula for the coefficients of the cyclotomic polynomial.*

A formula for the coefficients of the irreducible equation of the primitive nth roots of unity is derived. The formula is valid for any n. (Received February 1, 1938.)

103. Tomlinson Fort: *Theorems on the summability of series.*

Theorems for summability, analogous to the Abel-Dirichlet theorems in the theory of convergent series (see Fort, *Infinite Series*, Theorems 119 and 120), are developed in this paper. Borel summability is first considered and several general theorems, which are applied in detail to power series and Lambert series, are obtained. In the treatment of R-summability, general theorems, which are applied in detail to certain exponential series and to generalized factorial series, are proved. (Received January 26, 1938.)

104. J. S. Frame: *A symmetric representation of the* 27 *lines on a cubic surface by lines in a finite geometry.*

The group of isomorphisms of the 27 lines and 45 triple-tangent planes on a cubic surface is of order 51840 and contains the simple group of order 25920 as an invariant subgroup. The latter may be represented as a group of unitary transformations in a three-dimensional finite projective geometry, in which a plane is defined by four homogeneous coordinates from a modular field of four marks. A one-to-one correspondence between the 45 triple-tangent planes to the cubic surface and a certain set of 45 planes in the finite geometry is established, providing a simple criterion that two planes meet in a line of the surface. The 27 lines may be represented by those 27 number triples (a, b, c) which have just one coordinate equal to zero. Then the condition that two lines intersect on the surface is that they have just one coordinate in common. The third line of the triangle in which their plane cuts the surface is their vector sum (mod 2). Several configurations of lines and planes on the cubic surface are easily discussed in this representation, which seems, on account of symmetry, to be better adapted to their study than the familiar notation of Schläfli. Such, for instance, are the 40 triples of trihedral pairs, which are closely related to sets of those

40 planes of the finite geometry which do not correspond to triple-tangent planes on the cubic surface. (Received January 28, 1938.)

105. J. W. Givens: *Cayley parametrization and spin representation of orthogonal matrices.*

The spin representation of the complex orthogonal group leads to the association of two sets $\pm(\lambda, \lambda^{ij}, \lambda^{ijkl}, \cdots)$ of skew symmetric tensors of even rank with each proper orthogonal matrix $L=\|L^i{}_j\|$. The set contains one scalar and n tensors, where the order of L is $2n$ or $2n+1$. The components of these tensors are parameters which give an unexceptional representation of all proper orthogonal matrices. The relations connecting these tensors are found. One of them is $\lambda(L^i{}_j-\delta_j{}^i)=\lambda^{ir}\delta_{rs}(L^s{}_j+\delta_j{}^s)$, where $\|\delta_{rs}\|=\|\delta_s{}^r\|=1$. Hence $\lambda^{-1}\|\lambda^{ij}\|$ is the Cayley matrix $\Lambda=(L-1)(L+1)^{-1}$, which parametrizes all those orthogonal matrices for which $|L+1|\neq 0$. If the order of L is even and $|L-1|\neq 0$, the last two tensors in the set are related to the matrix $(L+1)(L-1)^{-1}$. These results are also obtained for the real orthogonal and Lorentz groups. (Received January 27, 1938.)

106. F. L. Griffin: *Discontinuous solutions for certain problems in the calculus of variations.*

In this note the author constructs, for certain new problems in the calculus of variations, discontinuous solutions analogous to those of Goldschmidt for the minimum area of revolution. Two of the problems considered relate to the surface of revolution which has the minimum moment of inertia with respect to the axis of revolution, or which exerts the minimum attraction upon a particle located on that axis. Continuous solutions of the several problems by extremals have been obtained in thesis studies by former students. (Received January 29, 1938.)

107. Israel Halperin: *Dimensionality in reducible geometries.*

The dimensionality theory of J. von Neumann, which applies to irreducible (classical projective and continuous) geometries, has been extended by the writer to reducible geometries with suitable sets of partial-automorphisms (see abstract 43-3-125). The present paper constructs an extensive class of reducible geometries to which this dimensionality theory applies. This is done by defining integration for functions whose values lie in an arbitrary continuous geometry. (Received January 28, 1938.)

108. Philip Hartman and R. B. Kershner: *On the Fourier transform of a singular function.*

The Fourier-Stieltjes transform $L(T;\sigma)=\int_{-\infty}^{+\infty}\exp(itx)d\sigma(x)$ of any distribution function $\sigma(x)$ is always $O(1)$. A necessary condition for $\sigma(x)$ to be absolutely continuous is that $L(T;\sigma)=o(1)$. Every known example of a distribution function $\sigma(x)$, which is not absolutely continuous but which satisfies $L(T;\sigma)=o(1)$, is almost everywhere constant. The authors give an example of a purely singular distribution function $\tau(x)$, such that the spectrum of τ is an entire interval, $L(T;\tau)=O(\log^{-1/2}|t|)$, and $L(t;\tau)\neq o(\log^{-1/2}|t|)$. The function τ is obtained as the infinite convolution of a sequence of simple step-functions. (Received January 27, 1938.)

109. Max Herzberger: *Partial differential equations and variation calculus.*

It is shown that the theory of partial differential equations and the geometrical problems in the calculus of variations can be considered as equivalent in many respects. A kind of Legendre transformation is used to transform problems from one region into problems of the other and vice versa. This concept allows the application of the results and methods found in one part to those of the other, especially to all of Hamilton's ideas. Simple geometric interpretations of the integration theories by Cauchy, Lagrange, Mayer, Natani, and Lie are thus obtained. It is also recognized why the theory of integral invariants must play an important rôle in both fields. The relation between the two fields becomes most significant if, in both problems, homogeneous coordinates are used. It is easy to develop the theory for the more complicated non-homogeneous case later on. (Received January 19, 1938.)

110. Edward Kasner and J. J. De Cicco: *The quadric fields in the geometry of the whirl-motion group G_6.*

This paper is a continuation of the paper, *The group of turns and slides and the geometry of turbines*, by Kasner, American Journal of Mathematics, vol. 33 (1911), and the abstract by the authors, *The geometry of the whirl-motion group G_6. Elementary invariants* (abstract 44-1-28). A *turbine* consists of the ∞^1 elements whose points are on a circle and whose directions make a constant angle α with the directions of the circle. A *flat field* consists of the ∞^2 elements co-circular with a fixed element. By means of a certain representation R whereby elements are represented by points of a three-space, it is found that the turbines and flat fields of the plane are pictured by the straight lines and the planes of space, respectively. In this representation, fields which correspond to the quadric surfaces of space are called *quadric fields.* It is found that *the quadric fields may be classified into nine distinct classes with respect to the whirl-motion group G_6.* Finally, the invariants for the quadric fields are found with respect to G_6. (Received January 10, 1938.)

111. J. L. Kelley: *A metric related to property S.*

It has been shown by Mazurkiewicz and by G. T. Whyburn that the metric for any connected, locally connected set may be redefined, homeomorphically, so that the neighborhood of a point is connected. As the solution to a problem proposed by G. T. Whyburn, a metric with the following properties has been derived: If the space is connected and locally connected, under the new definition of the metric, the neighborhood of a point is connected. If the space, furthermore, has property S, the ϵ neighborhood of any subset, for any ϵ, has property S. (Received January 29, 1938.)

112. D. H. Lehmer: *A factorization theorem applied to tests for primality.*

The writer has given several tests for primality, based on the converse of Fermat's theorem, which have been used during the last decade for the identification of many large primes. In order to apply any of these tests to the number N, some suitable divisor of $N-1$ must be known. For a general N, this requirement may be difficult to fulfill. However, the most interesting values of N are of the form $Q_n(a)$, where a is an integer, and $Q_n(x)$ is the irreducible cyclotomic polynomial whose roots are the primitive nth roots of unity, as, for example, the numbers of Fermat and Mersenne. This paper gives a theorem for the factorization of $Q_n(a)-1$ in terms of the factors of $Q_k(a)$ where $k<n$. There are three cases according as the number of odd prime factors of n is 0, 1, or more. (Received January 28, 1938.)

113. Howard Levi: *A characterization of the values assumed by polynomials.*

The difference quotient $f(z+h)/h-f(z)/h=k$ defines h as a function of z and k. If $f(z)$ is a polynomial of degree n, h is algebraic and its $n-1$ branches $h_1, \cdots, h_{n-1}$ satisfy $\sum_1^{n-1} h_i = -nz+\text{const.}$, $(n>2)$. It is also shown that if the difference quotient defines $n-1$ distinct functions $h_1, \cdots, h_{n-1}$, such that $\sum_1^{n-1} h_i = -nz+\text{const.}$, $(n>2)$, and if $f(z)$ is integral, then $f(z)$ is a polynomial of degree n. Both results are extended to functions of N variables. (Received January 17, 1938.)

114. A. N. Lowan: *On wave motion for infinite domains.*

The object of this paper is the integration of the non-homogeneous differential equation of wave motion $\nabla^2 u(P, t) = (a^2\partial^2/\partial t^2 + 2b\partial/\partial t)\, u(P, t) + \phi(P, t)$ where P is a point in the infinite space of one, two, or three dimensions, the solution satisfying the initial conditions: $\lim_{t\to 0} u(P, t) = f(P)$, $\lim_{t\to 0} \partial u(P, t)/\partial t = g(P)$. The paper is divided into four sections as follows: (1) one-dimensional non-resisting medium, (2) one-dimensional resisting medium, (3) two-dimensional resisting medium, (4) three-dimensional resisting medium. The method consists of reducing the given problem to one characterized by a non homogeneous ordinary differential equation for the function $u^*(P, p)$ which is the Laplace transform of the given function $u(P, t)$. The solution of the differential equation for $u^*(P, p)$ is obtained with the aid of a certain well known theorem in conjunction with the Fourier integral identities for one, two, and three dimensions. The transition from the solution for the u^*'s to the solutions for the u's is obtained with the aid of some standard methods in the operational calculus in conjunction with certain known identities involving infinite integrals whose integrands contain one or several Bessel functions. (Received January 20, 1938.)

115. N. H. McCoy: *Concerning matrices with elements in a commutative ring.*

In this paper, most of the fundamental theorems on matrices having to do with the characteristic function and the minimum function are suitably generalized to the case of matrices with elements in an arbitrary commutative ring R with unit element. If A is a matrix with elements in R, the ideal $\mathfrak{m}$ in $R[\lambda]$ of all polynomials $h(\lambda)$ such that $h(A)=0$ is the required analog of the minimum function, and a leading result is an explicit determination of this ideal. If $f(\lambda)$ is the characteristic function of A, then it is found that the ideals $(f(\lambda))$ and $\mathfrak{m}$ have the same prime ideal divisors, this being a generalization of the familiar theorem that the distinct linear factors of the characteristic function and of the minimum function are identical. Other generalizations of a similar nature are also obtained, one of them being an analog of a known theorem concerning the characteristic roots of a polynomial in several matrices, not necessarily commutative (cf. this Bulletin, vol. 42 (1936), pp. 592–600). (Received January 12, 1938.)

116. W. T. Martin and Norbert Wiener: *Taylor's series of functions of smooth growth in the unit circle.*

The authors are concerned with functions with positive coefficients analytic in the unit circle. By studying the behavior of a function along the positive real axis, relations among the coefficients are determined. The results are the best possible of this type. (Received January 8, 1938.)

117. C. B. Morrey: *The existence of solutions of certain minimum problems for multiple integrals.*

In this paper, minimum problems for non-parametric double integrals $I(z, G)$ (of $f(x, y, z, p, q)$ over G) are discussed where (1) G is a suitably restricted region, (2) (z, p, q) may stand for $(z', \cdots, z^n, p', \cdots, p^n, q', \cdots, q^n)$, (3) $f(x, y, z, p, q) \geqq 0$ everywhere and satisfies a uniform Lipschitz condition on any bounded portion of (x, y, z)-space, and (4) $f(x, y, z, p, q)$ is convex in (p, q) for each fixed (x, y, z). The class of "potential functions of their generalized derivatives" (a concept due to G. C. Evans) is used; such functions need not be continuous. Compactness theorems are proved for certain sets of these functions. It is shown that $I(z, G)$ is lower semicontinuous with respect to the type of convergence used in the compactness theorems. If there exist numbers $m>0$ and $\alpha>1$ such that $f(x, y, z, p, q) \geqq m(|p|^\alpha + |q|^\alpha)$, it is shown that a function z of our class exists which "takes on" (in a sense described in the paper) given continuous boundary values and minimizes $I(z, G)$ among all such functions. If also $\alpha>2$ or $\alpha=2$ and $f(x, y, z, p, q) \leqq M(p^2+q^2)$, the minimizing function is continuous on the closure $\bar{G}$. Analogous results hold for n-tuple integrals. (Received January 28, 1938.)

118. Tadasi Nakayama: *A note on the elementary divisor theory in non-commutative domains.*

The reduction of a matrix from a non-commutative domain of integrity to a van der Waerden-Wedderburn normal form (*Moderne Algebra*, vol. 2, 1932; Journal für Mathematik, vol. 167 (1932)), as well as to a Jacobson normal form (Annals of Mathematics, vol. 38 (1937)), can be achieved under the mere assumption that all right and left ideals are principal; one does not need to assume the so-called euclidean division process. Moreover, the diagonal elements in a Jacobson normal form are determined by the original matrix uniquely up to similarity. (Received January 14, 1938.)

119. B. J. Pettis: *On linear functionals and completely additive set functions.*

If T is an abstract space for which the generalized measure problem (Banach and Kuratowski, Fundamenta Mathematicae, vol. 14 (1929), p. 127) has a negative answer, then a sharper form can be given to the Lebesgue theorem concerning the decomposition of functions completely additive over an additive family of measurable sets; it is also possible to give the general form of the linear functionals over these functions. The extension can be made to the case of functions having their values in a linear normed complete space. (Received January 27, 1938.)

120. J. F. Randolph: *Linear equations in an infinite number of unknowns.** Preliminary report.

Certain special cases of a system of an infinite number of linear equations in an infinite number of unknowns have been studied. The method most closely related to that of a finite set of equations in a finite number of unknowns is perhaps that of von Koch, who obtains a unique bounded solution when the double array of coefficients of the unknowns is subjected to conditions symmetrical about the main di-

* This paper is chiefly the work of the late Professor D. C. Gillespie of Cornell University.

agonal and the constants form a bounded set. In this paper the same result is obtained, but the assumptions made concerning the coefficients are not symmetrical about the main diagonal. The fact that a system in which all coefficients above the diagonal are zero and those in the diagonal are not zero has a unique solution is cited as a partial explanation of this lack of symmetry. The conditions here neither imply nor are implied by those of von Koch. (Received January 27, 1938.)

121. L. B. Robinson: *On the analytic prolongation of a singular solution of the equation of Izumi.*

The author has found a singular solution $f(\lambda, x)$ of the equation of Izumi. Except at the origin, $f(\lambda, x)$ is holomorphic with respect to x. It also converges when $|\lambda| < 1$ and is meromorphic. By means of the formula $U_{p+1}^{(n)} = U_p^{(n)} + x(U_p^{(n)})^2/n$ due to Goursat, $f(\lambda, x)$ can be approximated by a series of polynomials. (Received February 3, 1938.)

122. J. B. Rosser and R. J. Walker: *The algebraic theory of diabolic magic squares.*

A square matrix of order n is said to admit a path $\{a, b\}$ if $(a, b, n) = 1$, and if, for all i and j, $\sum_{x=1}^{n} a_{i+ax, j+bx} = N$, the subscripts being reduced modulo n. The matrix is an algebraic diabolic square (a.d.s.) if it admits $\{0, 1\}$, $\{1, 0\}$, $\{1, 1\}$, and $\{1, n-1\}$; and an a.d.s. is a numerical diabolic square (n.d.s.) if the elements consist of the integers $1, 2, \cdots, n^2$ (so that $N = n(n^2+1)/2$). There is no n.d.s. of order 3, or of any order $\equiv 2 \pmod 4$. It is shown that there are n.d.s. of all other orders, and that one can construct at least $(n!)^2\,\theta(n)$ n.d.s. of order n by the classic "step-process," where $\theta(2^\alpha) = 0$, $\theta(p^\alpha) = p^{2\alpha-2}(p-3)(p-4)$ if p is an odd prime, and $\theta(mn) = \theta(m)\theta(n)$ if $(m, n) = 1$. If $n \not\equiv 2 \pmod 4$ and $\neq 3$ or 5, there exist n.d.s. of order n not constructible by the step-process. Hyper-a.d.s., which admit additional paths, are considered, and general (algebraic) solutions obtained. Using these, one can prove that a n.d.s. which admits all but two paths must be constructible by the step-process. A consequence of this is that all n.d.s. of order 5 are constructible by the step-process, and there are exactly 28,800 of them. Some of these results are extended to cubes. (Received January 28, 1938.)

123. O. K. Sagen: *A quasi-canonical form in the ring of integers.*

A non-singular integral quadratic form in n variables is reduced by a unimodular transformation to a unique form with the same leading coefficient. The reduction is a modification of that for the rational canonical form. Its interest lies in its application to the study of integer representations by quadratic forms in more than two variables. General results for a genus of ternary forms are readily obtained. (Received January 28, 1938.)

124. Wladimir Seidel and J. L. Walsh: *On the derivatives of functions analytic in the unit circle.*

Let $w = f(z)$ map conformally $|z| < 1$ onto a Riemann domain S. Let $R(w)$ denote the radius of the largest smooth circle on S with center w. If either $f(z)$ omits two values or $R(w)$ is bounded independently of w, then a necessary and sufficient condition for (1) $|f'(z_n)|(1-|z_n^2|) \to 0$, with $w_n = f(z_n)$ bounded, is $R(w_n) \to 0$. This condition is necessary but not sufficient for the general class of functions analytic for $|z| < 1$. The approach in (1) may be arbitrarily slow, even when $f(z)$ is bounded, univalent, and continuous for $|z| \leqq 1$. If $f(z)$ is univalent, one has $|f^{(k)}(z)|(1-|z|^2)^k$

$\leqq P_{k-1}(|z|)\cdot R[f(z)]$, where $P_{k-1}(|z|)$ is a polynomial in $|z|$ of degree $k-1$ with positive coefficients. For $k=1$ this inequality is due to Littlewood and Macintyre. If $f(z)$ is bounded, similar results can be obtained for $k=1$. If $f(z)$ is univalent, then at almost all points of the circumference $|f'(z)|(1-|z|)^{1/2}\to 0$, as z approaches the circumference angularly. (Received January 21, 1938.)

125. J. J. Stoker: *An eigenvalue problem in elasticity with continuous spectrum.*

A plane plate subjected to compressive or shear forces in the plane of the plate will buckle when such loads attain certain critical values, the determination of which leads to an eigenvalue problem involving a partial differential equation of fourth order. Two special cases have been found which lead to eigenvalue problems with continuous spectra—a phenomenon hitherto encountered in only one other physical problem (with a bounded domain) known to the author, that is, in the case of the Schrödinger equation in quantum mechanics. (Received January 27, 1938.)

126. J. J. Stoker: *On unbounded convex point sets.*

Let S^3 be an unbounded convex point set in three-dimensional euclidean space, $\bar{S}^3$ the set of its boundary points. It is shown that $\bar{S}^3$ is topologically one of the following forms: (1) the plane, (2) a pair of distinct planes, (3) an open cylinder. The spherical image of S^3 is shown to lie on a hemisphere; conditions under which the latter set is open or closed are given. (Received January 27, 1938.)

127. W. C. Strodt: *Sequences of systems of algebraic differential equations.*

Let $\Sigma_1, \Sigma_2, \cdots$ be a sequence of closed irreducible systems in the unknowns $y_1, \cdots, y_n$, such that the manifold of Σ_i is a proper part of the manifold of Σ_{i+1}, $(i=1, 2, \cdots)$. Let Σ be the set of forms common to the Σ_i. Then Σ is a closed irreducible system, and a necessary and sufficient condition for an ordered set $y_1(x), \cdots, y_n(x)$ of n functions analytic in a region $\mathfrak{B}$ to be a solution of Σ is that there exist a point x_0 in $\mathfrak{B}$ such that for every positive integer m and every positive number ϵ there is a positive integer i such that Σ_i has a solution $\bar{y}_1(x), \cdots, \bar{y}_n(x)$ analytic at x_0 with $|\bar{y}_{ij}(x_0)-y_{ij}(x_0)|<\epsilon$, $(i=1, \cdots, n; j=0, 1, \cdots, m)$, where the second subscript is an index of differentiation. When $y_1(x), \cdots, y_n(x)$ is a solution of Σ, the set of points x_0 consists of all the points in $\mathfrak{B}$ except for a set which is at most denumerable. (Received January 10, 1938.)

128. Otto Szász: *On the jump of a function determined by its Fourier series.*

Let $f(x)$ be integrable and of period 2π; hence it has a Fourier series. Denote by $\sigma_n(x)$ the arithmetic means of the partial sums of the series. It is well known that $\sigma_n(x)\to\{f(x+0)+f(x-0)\}/2$; to determine the jump $f(x+0)-f(x-0)=D(x)$, several devices have been used. The author gives a new and effective way for the solution of this problem. Denote by $\bar{\sigma}_n(x)$ the trigonometric polynomial conjugate to $\sigma_n(x)$; the result is $D(x)=\lim_{n\to\infty}(\pi/\log 2)\{\bar{\sigma}_{2n}(x)-\bar{\sigma}_n(x)\}$, whenever $D(x)$ exists in the usual sense or as an integral mean. (Received January 27, 1938.)

129. A. E. Taylor (National Research Fellow): *Linear differential systems and linear operations analytic in a parameter.*

For each complex t in the circle $|t|<r$ let A_t be a bounded linear transformation

in the complex Banach space E, and let A_tx be analytic (differentiable) for each $x\epsilon E$. The author considers the differential systems $dx/dt=A_tx$, $x(0)=y$ and $dx/dt=A_tx+z(t)$, $x(0)=0$, where y is arbitrary in E and $z(t)$ is analytic, $|t|<r$, with values in E. The unique analytic solution of the first system is given by $x=B_ty$, where B_t is a linear operation analytic in t (as A_t is). For each t, $|t|<r$, B_t defines a one-to-one mapping of E on itself, with analytic inverse B_t^{-1}. A solution of the non-homogeneous system is then $x=B_t\int_0^t B_s^{-1}z(s)ds$. In developing the theory use is made of an "adjoint" system involving the conjugate operation A_t^* defined in the linear functional space conjugate to E. The results include many of the known theorems on differential systems in an infinite number of unknowns, as well as the classical theory. (Received January 24, 1938.)

130. A. E. Taylor (National Research Fellow): *Linear operations which depend analytically on a parameter.*

Let E, E' denote complex Banach spaces. Let A_t be a bounded linear operation with domain E and range in E' for each t in a region Δ of the complex plane. If for each $x\epsilon E$, A_tx is differentiable as a function on Δ to E', it is said that A_t depends analytically on t. It is then also analytic (differentiable) as a function on Δ to the space of bounded linear operations on E to E' when this space is normed by the bound of the operation. Application of this result to linear functionals yields results of the following sort: In order that a sequence $\{x_n(t)\}$ of numerical functions define an analytic function on Δ to (l_p), $(p\geqq 1)$, it is necessary and sufficient that (i) $x_n(t)$ be analytic in Δ, (ii) $\sum_1^\infty |x_n(t)|^p<\infty$ in Δ, and the series be bounded in each bounded closed subregion of Δ. Corresponding theorems are obtained for the spaces (c), (m), L_p, $(p\geqq 1)$. The set of points in Δ for which A_t admits a bounded inverse with domain E' is either null or open, and in the latter case the inverse is analytic also. (Received January 24, 1938.)

131. S. M. Ulam: *On bounded transformations of spaces.* Preliminary report.

A continuous transformation T of a metric space into itself will be said to have the bound k, if k is the g.l.b. of numbers l such that $\rho(T^n(p),p)\leqq l$ for all p and n. A class of locally connected continua C is determined, such that there exist positive numbers $k(C)$ with the property that any T transforming C into itself and having a bound $<k(C)$ must have a fixed point $T(p_0)=p_0$. (Received January 27, 1938.)

132. S. M. Ulam: *Set-theoretical invariants of the product operation.*

Let K be a class consisting of countably many subsets A_n of a set E (for example, rational intervals on the line). Let $B(K)$ be the smallest class of sets containing K and closed with respect to the operations of set-difference and countable addition. Properties of classes which can be imbedded in a $B(K)$, in particular Borel classifications, are established. Let A be an abstract set and L the class of all subsets of $A^2=A\times A$ of the form $A_1\times A_2$, $A_1, A_2\subset A$. Properties of a Borel classification of $B(L)$ are determined, in particular invariance with respect to product-isomorphisms (transformations of the form $(a_1, a_2)\rightarrow(f(a_1), f(a_2))$, where f is an arbitrary one-to-one transformation of A into itself). (Received January 27, 1938.)

133. Morgan Ward: *The law of apparition of primes in a Lucasian sequence.*

A sequence $u_0, u_1, u_2, \cdots$ of rational integers is said to be Lucasian if it satisfies a linear recursion relation with constant integral coefficients and if u_n divides u_m whenever n divides m. The author considers Lucasian sequences of a form sufficiently general to include all heretofore published instances and reduces the problem of determining à priori all terms of such a sequence divisible by a preassigned prime modulus to the fundamental problem of determining the period of a mark in a finite field. This paper will appear shortly in the Transactions of this Society. (Received January 13, 1938.)

134. Hassler Whitney: *Tensor products of abelian groups.*

Given two abelian groups G and H, let S be the set of all finite "sums" $g_1 \cdot h_1 + \cdots + g_n \cdot h_n$ (any n). If any two elements of S which may be proved equal with the help of the two distributive laws are identified, one obtains an abelian group, the *tensor product* of G and H. Fundamental properties are studied, including relations with operator rings. If G and H are linear, it is assumed also that $rg \cdot h = g \cdot rh$ (real r). If the groups are topological and separable, a topology may be introduced into the product. The elements of tensor analysis, including contraction and covariant differentiation, are simply expressed with the help of tensor products (without using coordinate systems). (Received January 24, 1938.)

135. Leonidas Alaoglu: *Weak convergence of linear functionals.*

The definition of weak convergence of sequences of linear functionals given by Banach (*Opérations Linéaires*, p. 122, §4) is extended by the use of the Moore-Smith limit. A directed set $(f_\alpha | \alpha)$ of linear functionals on a space X of type B is defined to be weakly convergent to the functional f if the set is bounded, and if $\lim_\alpha f_\alpha(x) = f(x)$ for every x in X. The topology so defined in the adjoint space X^* is equivalent to a Hausdorff topology in the bounded subsets of X^*, and is such that weakly closed and bounded subsets are bicompact. By means of this result it is shown that a linear subspace of X^* is regularly closed if and only if it is weakly closed. The other theorems of Banach (chap. 8) which refer to weak convergence are similarly extended, the condition that X be separable being removed. (Received February 1, 1938.)

136. A. A. Albert: *On cyclic algebras.*

Let Z of order n over K be a direct sum of t equivalent cyclic fields over K, so that Z has an automorphism S of order n over K. Define a cyclic system (Z, S) of all pairs Z', S' with Z' equivalent to Z under a correspondence such that S' goes into S. Such systems were considered by O. Teichmiller, Deutsche Mathematik, vol. 1 (1936), pp. 197–238. He connected them with corresponding cyclic algebras (Z_L, S, x) over $L = K(x)$, and used the known results on cyclic algebras to prove a number of corresponding theorems on the cyclic systems. The author proves the theorems on the cyclic systems by elementary commutative arguments and uses the results to provide new proofs of the theorems on cyclic algebras. In particular let Z be a cyclic field, d be a divisor of n, Y be the subfield of Z of degree nd^{-1}. Then the present work provides the first simple proof of the theorem $(Z, S, \gamma)^d \sim (Y, S, \gamma)$. (Received March 1, 1938.)

137. C. R. Cassity: *The maps determined by the principal curves associated with five and six points in the plane.*

The group associated with five points is of order 1920. The map has twenty four-sided and sixteen five-sided regions. It has only partial symmetry, since only one hundred and sixty elements can be interpreted as topological transformations of the map into itself. A study of the map obtained on the cubic surface by the cubic curves on six points of the plane is the principal purpose of the present paper. The case which leads to twenty-seven real lines with no three concurrent is that under consideration. The number and types of regions were given by Klein (Mathematische Annalen, vol. 6 (1873), p. 570) and Zeuthen (Mathematische Annalen, vol. 8 (1875), pp. 1–30). The author shows that there is only one type of map and that it has no symmetry whatever. The surface can be put onto itself in only the identical way. The statement of Zeuthen that any of the fifteen lines, each of which forms the boundary of two triangles, can be "shifted across" either triangle independently of the other is examined and a method is derived for determining the number of other "shifts" which must precede any desired one. The methods are largely synthetic. (Received February 23, 1938.)

138. Harold Chatland: *A note on the asymptotic Waring problem for homogeneous polynomial summands.*

In a paper appearing in the January, 1938, issue of the Annals of Mathematics, the author proved that the number of homogeneous polynomials of the form $\sum_{r=0}^{n} a_r x^{n-r} y^r$ which suffices to express integers greater than a certain (large) integer is about two-thirds of the number of nth powers required. By a new treatment of some inequalities appearing in the paper mentioned, the number of homogeneous polynomial summands required to express such integers has been reduced to about one-half the number of nth-power summands. The restriction $a_0 = a_1 = a_2 = 1$ on the integral coefficients is retained in this paper. (Received February 24, 1938.)

139. J. M. Dobbie: *A generalized Lambert series.*

Let $h(x)$ be a function which is analytic interior to $|x| = 1$ and which has the value 1 at $x = 0$. Then the series $\sum_1^\infty b_n x^{\lambda n} h(x^n)$, ($\lambda$ a positive integer), is called a generalized Lambert series and is discussed as to its convergence properties, necessary and sufficient conditions for expanding a function in such a series, and the existence of a natural boundary for the sum function. On this latter subject are proved theorems analogous to the theorems for ordinary Lambert series proved by Knopp (Journal für die reine und angewandte Mathematik, vol. 142 (1913), pp. 283–315) and by Hardy (Proceedings of the London Mathematical Society, (2), vol. 13 (1913), pp. 192–198). The hypotheses on the coefficients b_n are stated in terms of Cesàro summability for a non-negative integer, while the approach is radial. (Received March 1, 1938.)

140. Nelson Dunford: *One parameter groups of linear transformations.*

A one parameter group T_s, $(-\infty < s < \infty)$, of linear transformations on a Banach space X is continuous in the strong topology of the ring of operators on X if: (i) $\|T_s\|$ is bounded in some interval about $s = 0$; (ii) for each x, the set $T_s x$, $(-\infty < s < \infty)$, is separable; and (iii) for each y in $\overline{X}$ and x in X, the function $yT_s x$ is measurable in the sense of Lebesgue. (Received February 26, 1938.)

141. Arnold Emch: *New properties of the cubic surface.*

In 1931 the author found a new normal form of the cubic surface which was published in the American Journal of Mathematics, vol. 53, pp. 902–910. This form permits one to prove algebraically many properties of the cubic which up to the present time had been established only by synthetic methods. Moreover a new connection is shown between the Δ_{18}-configuration of syzygetic pencils and the Steiner-Dixon summit-planes of associated triples of couples of conjugate Steinerian trihedrals. It is shown that a general cubic surface can be put into this normal form in at least forty different ways; in one way at least for each of the forty summit-planes. (Received February 18, 1938.)

142. Aline H. Frink and Orrin Frink: *Polygonal variations.*

Arcs minimizing an integral $\int f(x, y, y')dx$ may exist even though many of the partial derivatives of $f(x, y, y')$, which occur in the usual treatment of the calculus of variations, do not exist. Hence it is important to have proofs of the familiar necessary conditions which require the existence of as few partial derivatives of f as possible. It is shown in this paper that, by giving the dependent variable variations whose graphs are polygonal lines of proper shape depending on a parameter ϵ, and by assuming the existence of only the single partial derivative $f_{y'}$, it is possible to derive the Weierstrass necessary condition and a generalization of the Euler equation. The ordinary form of the Euler equation can be derived by this method without the use of integration by parts or Du Bois Reymond's lemma. A slight generalization of the Legendre necessary condition is also obtained, assuming the existence of only the single generalized second partial derivative $f_{y'y'}$. The method extends to the case of more than one dependent variable, and to the parametric problem. (Received February 24, 1938.)

143. Einar Hille: *On semi-groups of transformations in Hilbert space.*

If $T_\alpha x$ is a family of self-adjoint, positive definite transformations in Hilbert space, defined for $\alpha > 0$ and having the semi-group property $T_\alpha[T_\beta] = T_{\alpha+\beta}$, and if $\|T_\alpha x\| \leqq \|x\|$, then $(T_\alpha x, y) = \int_0^\infty e^{-\alpha\lambda} d(E(\lambda)x, y)$. Here $E(\lambda)$ is the resolution of the identity of a self-adjoint, positive definite transformation $Ax = (-\log T_1)x$. In particular, $(T_\alpha x, y)$ is an analytic function of α, holomorphic and bounded for $\Re(\alpha) > 0$. The author has determined the transformation A for the transformations associated with the Gauss-Weierstrass and the Poisson singular integrals. (Received February 26, 1938.)

144. Einar Hille: *On the absolute convergence of polynomial series.*

If the terms of a series are polynomials with real zeros, the absolute convergence of the series at a non-real point implies absolute convergence in a certain region. Using elementary methods, the author determines this region under various assumptions on the distribution of the zeros. (Received February 26, 1938.)

145. Charles Hopkins: *The half-group of cosets belonging to a group.*

Let G denote any group; let Ω denote a set of operators for G which contains at least all the inner automorphisms of G; and let S denote the set of all admissible subgroups H of G. Each H gives rise to the quotient group $\Gamma = G/H$, and each Γ is isomorphic with the group of cosets of G with respect to H. Under the usual definition of multiplication for complexes the set of all distinct cosets of G with respect to all elements of S will form a half-group $\Gamma(G)$. Certain properties of $\Gamma(G)$ are discussed. For the case when S contains a finite number n of elements it is shown that the "extended group-ring of G," that is, the hypercomplex system $R(G)$ (over a field F) which has as a basis the elements of $\Gamma(G)$, is the direct sum of n invariant subrings $\overline{R}_1 + \cdots + \overline{R}_n$, where each $\overline{R}_i$ is ring-isomorphic with the regular group-ring of Γ_i over F. (Received March 1, 1938.)

146. F. B. Jones: *Concerning R. L. Moore's Axiom* 5.

Consider the following axiom: Axiom 5_0. *If A is a point of a region R and B is a point distinct from A, there exists in R a compact continuum separating A from B.* If a space satisfies Axioms 0–4 of R. L. Moore's *Foundations of Point Set Theory* and Axiom 5_0, it will also satisfy Moore's Axiom 5. (Received February 25, 1938.)

147. H. L. Krall: *On certain differential equations for Tchebycheff polynomials.*

The purpose of this paper is to find conditions on the coefficients $\{l_{ij}\}$ in order that the differential equation $\sum_{i=0}^{r}(\sum_{j=0}^{i} l_{ij}x^j)y_n^i(x) = \lambda_n y_n(x)$, ($\{l_{ij}\}$ constants, λ_n a parameter), will have a set of orthogonal polynomials as solutions. Conditions are found which are necessary and sufficient for the existence of such a differential equation. The classical polynomials of Jacobi, Hermite, and Laguerre give us examples of polynomials which satisfy such equations. An example of a non-classical set of orthogonal polynomials is given which also satisfies an equation of the above type. (Received February 26, 1938.)

148. O. E. Lancaster: *Sequences defined by non-linear algebraic difference equations.*

This paper treats of sequences defined by difference equations of the form $\Phi(s_{n+m}, s_{n+m-1}, \cdots, s_n, n) = 0$, where Φ is a polynomial with real rational coefficients in its arguments $s_{n+m}, s_{n+m-1}, \cdots, s_n, n$, and n is an integer. The considerations are confined to those sequences which approach finite limits. Some criteria, sufficient to insure that the sequences defined by a given difference equation approach a constant limit α, are developed. A study is made of the relation between the rate of convergence of the sequences and the order of the difference equation, and between the rate of convergence of the sequences and the degree of n in the coefficients of the difference equation. Special attention is given to first order difference equations whose rational sequences approach $\beta^{1/p}$, where p is an integer, β is a rational number, and $\beta^{j/p}$, $(j = 1, 2, \cdots, p-1)$, is an irrational number. (Received February 23, 1938.)

149. Saunders MacLane: *Transcendence degrees and p-bases in terms of non-modular lattices.*

R. Baer has remarked that the invariance of the transcendence degree of a field and of the rank of certain abelian groups should have a common lattice-theoretic origin. This paper analyzes the type of lattice concerned: a continuous lattice satisfying a point-exchange axiom, a finiteness axiom, and an axiom insuring the existence of sufficient points. In such a lattice a "basis" of independent points can always be constructed. Its cardinal number is an invariant. This result includes the two invariants above and yields a new theorem for the p-basis of an inseparable algebraic extension of a field of characteristic p. The crucial point-exchange axiom in the lattice is weaker than the usual modular law. For other applications, this exchange axiom can be stated in two alternative forms, which, like the modular law, involve no reference to points: (I), $c > d = (a+b) \cdot c$ and $ab \geqq d$ imply the existence of a c_1 with $d < c_1 \leqq c$ and $ab = (a+c_1)b$; (II), $b \leqq a+c$, b not $\leqq c$ imply the existence of c_1 and c_2 with $c_1 + c_2 \leqq c$, $0 < c_1 \leqq a+b+c_2$, and b not $\leqq a+c_2$. (Received February 11, 1938.)

150. P. T. Maker: *Conditions on $u(x, y)$ and $v(x, y)$ necessary and sufficient for the regularity of $u+iv$.*

This paper extends the Looman-Menchoff theorem as follows: Let $\mathfrak{F}$ be a class of continuous functions defined in the open set G in the complex plane, and let $\mathcal{K}$ be a class of G for which regularity of any function of $\mathfrak{F}$ in $G-E$, $(E\varepsilon K)$, implies its regularity in G. If $f(z)$, $[=u(x, y)+iv(x, y)]$, $\varepsilon\mathfrak{F}$, and the Dini partial derivates of u and v are infinite on $\sum_{n=1}^{\infty}E_n$, $(E_n\varepsilon\mathcal{K}$, and closed in $G)$ at most, with the Cauchy-Riemann equations holding almost everywhere where the partials exist, $f(z)$ is regular in G. When u and v are of bounded variation (Tonelli), it is shown that if almost all parallels to the axes intersect E in a denumerable set at most, then $E\varepsilon\mathcal{K}$. For u and v absolutely continuous, $f(z)$ is shown to be regular if the Cauchy-Riemann condition stated above holds. It follows that $f(x, y)$ is harmonic if and only if $f(x, y)$ is absolutely continuous and $\int(df/dn)ds=0$ around "almost all" rectangles in G. (Received February 28, 1938.)

151. J. D. Mancill: *On volumes bounded by cylindrical surfaces.*

The problem with which this note is concerned is that of determining certain volumes common to two given cylindrical surfaces. The usual procedure of integral calculus leads to extremely difficult integration even in the simplest problems unless the cylindrical surfaces have their elements perpendicular. It is shown in this paper how these procedures may be slightly altered to simplify greatly the integration in the general case. Two very interesting properties of cylindrical surfaces are also derived. This note is prompted by repeated requests for solutions of problems of this nature. (Received February 12, 1938.)

152. F. D. Murnaghan: *The generalized Clebsch-Gordan series.*

The Clebsch-Gordan series furnishes the analysis of the Kronecker product (a) of irreducible representations of the two-dimensional unimodular linear group and (b) of irreducible representations of the three-dimensional rotation group. It is shown here that (a) is a special instance of the formula: $D(\lambda)\times(D(\mu)=D(\lambda, \mu)+D(\lambda+1, \mu-1) + \cdots +D(\lambda-\mu)$ which is valid for the n-dimensional full linear group; and that (b) is a special instance of the formula: $D(\lambda)-\times D(\mu)=\{D(\lambda_1+\lambda_2)+D(\lambda_1+\lambda_2-2)+\cdots +D(\lambda_1-\lambda_2)\}+\{D(\lambda_1+\lambda_2-1,1)+D(\lambda_1+\lambda_2-3,1)+\cdots+D(\lambda_1-\lambda_2+1,1)\} +\{D(\lambda_1+\lambda_2-2, 2)+D(\lambda_1+\lambda_2-4, 2)+\cdots +D(\lambda_1-\lambda_2+2, 2)\}+\cdots+D(\lambda_1, \lambda_2)$; for example, $D(3)\times D(2)=D(5)+D(3)+D(1)+D(4, 1)+D(2, 1)+D(3, 2)$ which is valid for the full n-dimensional real orthogonal group. When $n=3$ or 2, the $D(\mu_1, \mu_2)$ which depend on two labels must be modified according to the following "modification" rules. When $n=2$: all $D(\mu_1, \mu_2)$ for which $\mu_2>0$ are dropped; all $D(\mu_1, 2)$ are replaced by $-D(\mu_1)$; all $D(\mu_1, 1)$ for which $\mu_1>1$ are dropped; $D(1, 1)$ is replaced by $D^*(0)$, the star denoting the associated representation. When $n=3$: all $D(\mu_1, \mu_2)$ for which $\mu_2>1$ are dropped; all $D(\mu_1, 1)$ are replaced by $D^*(\mu_1)$. For the rotation subgroup of the full real orthogonal group, the stars may be dropped. (Received February 24, 1938.)

153. F. D. Murnaghan: *The analysis of the representations of the real n-dimensional orthogonal group induced by irreducible, rational, integral representations of the full linear group.*

Denoting by k the integer $n/2$ or $(n-1)/2$ according as n is even or odd, one can give the analysis of the representation $D(\lambda_1, \cdots, \lambda_j)$, $(j=1, 2, \cdots, k)$, by the formula $D(\lambda_1, \cdots, \lambda_j) = [\prod(1-\xi_p\xi_q)^{-1}]\cdot\Delta(\lambda_1, \cdots, \lambda_j)$, in which p and q vary from 1 to j, $p \leqq q$, and $j=1, \cdots, k$. In the product, ξ_p is an operator which reduces the label attached to λ_p by unity, and ξ_q is a similar operator. The Δ are irreducible representations of the orthogonal group; for example, $D(\lambda_1)=\Delta(\lambda_1)+\Delta(\lambda_1-2)+\Delta(\lambda_1-4)+\cdots$; $D(4, 2)=\Delta(4, 2)+\Delta(4)+\Delta(3, 1)+\Delta(2^2)+2\Delta(2)+\Delta(0)$. The above formula is also valid when $j>k$, but in such cases the $\Delta(\lambda_1, \cdots, \lambda_p)$, $p>k$, must be "modified" in accordance with "modification rules" given in the preceding abstract for $j=k+1$. For example, $n=2$, $D(4, 2)=\Delta(2)+\Delta(0)$; $n=3$, $D(4, 2)=\Delta(4)+\Delta^*(3)+2\Delta(2)+\Delta(0)$, the star denoting the associated representation. The method here described is available (and simpler in application) for the complex group. (Received February 24, 1938.)

154. W. T. Puckett: *Concerning local connectedness under the inverse of certain continuous transformations.*

A continuous transformation $T(A)=B$ on the compact set A is said to have property β relative to (A, B) provided that for $\epsilon>0$ there exists a $\delta>0$ such that every pair of points, $p,q \varepsilon T^{-1}(b)$, $(b\varepsilon B)$, with $\rho(p, q)<\delta$ lie in a connected subset of A of diameter $\leqq\epsilon$. In case A is locally connected, K is any closed subset of B with $F(B-K)$ finite, and $T^{-1}(b)$, $(b\varepsilon B)$, is locally connected, then T has property β relative to $[T^{-1}(K), K]$. Suppose T interior and B locally connected. Then in order that A be locally connected it is necessary and sufficient that T have property β relative to (A, B). In case A is a continuum and $T(A)=B$ is 0-regular (see A. D. Wallace, abstract 44-3-161), T has property β relative to (A, B). Thus local connectedness is invariant under the inverse, since T is also interior. If T is continuous and the inverse of every point is finite (countable), then the property of being a regular (rational) curve is invariant under the inverse. In case T is interior and finite, local connectedness is invariant under the inverse. (Received February 26, 1938.)

155. W. T. Puckett: *Concerning the* 1*-dimensional Betti group under interior transformations.*

Let $T(A)=B$ be an interior transformation on the compact set A. Let N be a subcontinuum of A such that $N\cdot\overline{(A-N)}=Q$ is connected and $p^1[T(Q)]=0$. Then for $\epsilon>0$ there exists a $\delta>0$ such that if Z is any 1-dimensional rational δ-cycle in $T(N)$, there exists a 1-dimensional ϵ-cycle C in N such that $f(C)\underset{\epsilon}{\sim}Z$. Consequently, if V is any rational Vietoris (or true) cycle in $T(N)$, there exists a Vietoris (or true) cycle W in N such that $f(W)\underset{\epsilon}{\sim}V$. This is a strengthening of a theorem by G. T. Whyburn (in a paper to appear in an early number of the Duke Mathematical Journal). From this paper it follows that if $p^1[T(N)]$ is finite the group $B^1_{\mathrm{R}}(N)$ maps homomorphically onto $B^1_{\mathrm{R}}[T(N)]$, and $p^1[T(N)]=p^1(N)$. (Received February 26, 1938.)

156. M. S. Robertson: *Piecemeal univalency of analytic functions.*

The properties of functions $f(z)$, $f'(0)=1$, which are regular and univalent within the unit circle have been studied in detail by many authors. In this paper, a more general class of functions $f(z)=z^k+\sum_{k+1}^{\infty}a_nz^n$, regular in the unit circle, is considered. These functions are not necessarily univalent within the unit circle as a whole, but only in pieces of the circle consisting of all the sectors of the circle having an aperture $\alpha\pi$, where $0<\alpha\leqq 2$. For this class of functions, which contains as a subclass the univalent functions mentioned above, it is shown that $|f(re^{i\theta})|\leqq A(\alpha)r^k(1-r)^{-2}$,

$(1/(2\pi))\int_0^{2\pi}|f(re^{i\theta})|d\theta<[2/\alpha]A(\alpha)r^k(1-r)^{-1}$, $|a_n|<[2/\alpha]A(\alpha)en$; wherein $A(\alpha)$ is a constant depending upon α, but not upon $f(z)$. (Received February 23, 1938.)

157. W. E. Sewell: *Degree of approximation by polynomials in z and 1/z.*

The theorem of Marcel Riesz (Jahresbericht der Deutscher Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368) on the modulus of the derivative of a polynomial in z and $1/z$ on the unit circle is extended to generalized derivatives on Jordan curves. These results are applied to the approximation of a function defined on a Jordan curve. (Received February 17, 1938.)

158. W. E. Sewell: *Note on the relation between Lipschitz conditions and degree of polynomial approximation.*

Let E, with boundary C, be a closed limited convex set and let $f(z)$ be defined on C. For each n, $(n=1, 2, \cdots)$, let there exist a polynomial $P_n(z)$ of degree n in z such that $|f(z)-P_n(z)|\leqq M/n^\alpha$, $\alpha>0$, z on C, where M is a constant independent of n and z. Then $f(z)$ satisfies on C a Lipschitz condition of order $\alpha/(t+\alpha)$, where t is a constant depending only on C. (Received February 17, 1938.)

159. R. M. Thrall: *Apolarity of trilinear forms and pencils of bilinear forms.*

In this note the idea of apolar trilinear forms, introduced earlier by the author (*Metabelian groups and trilinear forms*, American Journal of Mathematics, vol. 60 (1938)), is generalized and the main theorems are given new proofs independent of the group theoretic methods and notation used then. The importance of the apolarity concept in trilinear form classification is illustrated by a numerical application. In §3 there is given a new method of classifying singular pencils of bilinear forms, based on Dickson's minimal numbers (*Singular case of pairs of bilinear, quadratic or hermitian forms*, Transactions of this Society, vol. 29 (1927), pp. 239–253). These numbers are given by specific formulas in terms of ranks of constant matrices. (Received February 23, 1938.)

160. A. D. Wallace: *On local homeomorphisms.*

If $T(A)=B$ is a local homeomorphism (S. Eilenberg, Fundamenta Mathematicae, vol. 24 (1935), p. 35) on the locally connected continuum A, the following results are established: (1) If H is a simple closed curve in B, every component of the inverse of H is a simple closed curve, and T is topologically equivalent to $w=z^k$ on each of them. (2) If D is a dendrite in B, each component of the inverse of D is a dendrite and T is topological on each of them. (3) If H is an A-set in A, $T(H)$ is an A-set in B. (4) If x is a cut point in A, $T(x)$ is a cut point in B. (5) If A is a linear graph and X is the Euler characteristic, $X(A)=kX(B)$, where k is the multiplicity of T. (Received February 26, 1938.)

161. A. D. Wallace: *On 0-regular transformations.*

If A is a metric space and $T(A)=B$ is continuous, then T is said to be *0-regular* provided the convergence of the sequence of points x_n to the point x in B implies the 0-regular convergence of the sets $T^{-1}(x_n)$ to $T^{-1}(x)$ in A. (For the definition of 0-regular convergence see G. T. Whyburn, Fundamenta Mathematicae, vol. 25 (1935), p. 412.) Under the assumption that A is a compact continuum, the following theorems are proved: (1) For each x in B, $T^{-1}(x)$ is locally connected. (2) $T=T_2T_1$, $T_1(A)=A'$, $T_2(A')=B$, where T_1 is monotone and 0-regular and T_2 is finite, of constant multiplicity, and 0-regular. (3) If x is a cut-point of A then $T(x)$ is a local separating point of B. (4) T is 0-regular on open sets in A and on each component of the inverse of a connected set. (Received February 26, 1938.)

162. A. D. Wallace: *On regular transformations.*

The continuous transformation $T(M)=N$, on the compact metric space M, is said to have property Z (Wilson, American Journal of Mathematics, vol. 54 (1932), p. 377) if for each x in N and each positive ϵ there is a positive δ and a continuum K in M containing the δ-neighborhood of $T^{-1}(x)$ and contained in the ϵ-neighborhood of $T^{-1}(x)$; T is monotone if the inverse of every point is connected. T is said to be *regular* (Wilson, loc. cit.) if it is monotone and has property Z. In this paper it is shown that a necessary and sufficient condition that T be regular is that it have property Z and that N be locally connected. (Received February 26, 1938.)

163. F. P. Welch: *Non-linear singular differential equations containing a parameter.*

This work is an investigation of the properties of solutions of the equation $y'(x, \lambda)=\lambda^p a(x, \lambda, y)$, p an integer >0, (λ a complex parameter, x on the interval (a, b)). Here $a(x, \lambda, y)$ is analytic in λ and y and indefinitely differentiable in x at the point $\lambda=\infty$, $y=0$, x on (a, b). The solutions have been studied in the neighborhood of the point $x=\lambda$. The results of W. J. Trjitzinsky's paper in Acta Mathematica, vol. 66, were used for working out integrations involved in this paper. (Received March 1, 1938.)

164. G. T. Whyburn: *Interior surface transformations.*

Let $T(A)=B$ be a light interior transformation where A (hence also B) is a compact 2-dimensional manifold. Let α and β be the boundaries (if any) of A and B respectively. Let X be the set of all points of $T^{-1}(\beta)$ of order $\neq 2$, let Y be $T(X)$ plus all points of $B-\beta$ whose inverse contains points at which T is not locally topological, and let k be the multiplicity of T on the set $A-T^{-1}(Y)-T^{-1}(\beta)$. Let r and n be the numbers of points in Y and $T^{-1}(Y)$ respectively, and let m be the number of components of $T^{-1}(\beta)-\alpha-T^{-1}(Y)\cdot T^{-1}(\beta)$ which are not simple closed curves. Then if χ denotes the Euler characteristic, it is shown that $k\chi(B)-\chi(A)=kr-n-m$. Among the many consequences of this result the following are mentioned: (1) if $m=0$ (as, for example, when $\alpha=T^{-1}(\beta)$) and $\chi(A)=\chi(B)$, then either T is topological or else $r\geqq\chi(A)$. (2) If $0>\chi(A)=\chi(B)$ and $m=0$, the transformation is necessarily topological. (3) If $rk=n$ and $m=0$ (for example, if T is locally topological), then (i) $\chi(A)$ and $\chi(B)$ vanish or fail to vanish together and (ii) if $\chi(A)=\chi(B)\neq 0$, T is necessarily topological. (Received February 28, 1938.)

165. John Williamson: *Matrices normal with respect to an hermitian matrix.*

A square matrix A, with elements in the complex number field, is defined to be normal with respect to a non-singular hermitian matrix H if $AH=Hf(A^*)$, where $f(A^*)$ is a polynomial in the conjugate transpose of A. The theory of matrices normal with respect to H is similar to that of matrices normal with respect to any matrix S conjunctively equivalent to H. When S is suitably chosen, canonical forms for matrices normal with respect to S, under similar transformations by matrices which are conjunctive automorphs of S, are determined. By specializing $f(A^*)$ to be equal to A^*, the known results on the theory of the conjunctive equivalence of two non-singular pencils of hermitian matrices are obtained. (Received February 25, 1938.)

166. H. S. Zuckerman (National Research Fellow): *On the coefficients of certain modular forms belonging to subgroups of the modular group.*

A modular function $F(\tau)$ belonging to the full modular group has a single Fourier expansion about the parabolic point $\tau = i\infty$. In the case of modular functions belonging to a subgroup, however, it is necessary to consider expansions about a set of non-equivalent parabolic points. The coefficients of these expansions are obtained by a method (developed by H. A. Rademacher and H. S. Zuckerman in a recent paper) which is a modification of the Hardy-Ramanujan method. Applications to the reciprocals of the theta-functions are given. (Received February 26, 1938.)