ALBERT ON MODERN HIGHER ALGEBRA

Modern Higher Algebra. By A. A. Albert. (The University of Chicago Science Series.) Chicago, The University of Chicago Press, 1937. 14+319 pp.

Since the time of Dedekind it has become increasingly evident that the most satisfactory way of dealing with algebraic questions is to found the whole structure of algebra abstractly on certain fundamental abstract concepts. These are the concepts of group, ring, and field. The book we are considering here is apparently the first one in English to carry out this program. The contents of the book may be briefly summarized as follows: 1. Foundations; theory of groups, rings, fields (Chapter 1, 2). 2. Matrices and matrix algebras (Chapters 3, 4, 5, 10). 3. Galois theory (Chapters 6, 7, 8, 9). 4. Theory of valuation (Chapters 11, 12).

1. The book begins with a discussion of groups and their simpler properties and then goes on to define what is called a *ring*. A ring is an additive group in which multiplication is defined so as to be associative (but not necessarily commutative) and distributive with respect to addition. A fundamental idea here is that of *equivalent* groups and *equivalent* rings, the term "equivalent" being synonymous with the more commonly used "isomorphic." The most important concept is the construction of the polynomial ring in a scalar indeterminate over a given ring; the treatment of this subject on pages 18–19 seems somewhat confusing, although it is fundamentally sound.

Next are treated special kinds of rings, notably *integral domains*, which are commutative rings with an identity element and no zero divisors, and *fields*, which are integral domains in which division is always possible (except by zero). Every integral domain is contained as a subring in a field; the simplest example of this is the integral domain of ordinary integers which is contained in the field of rational numbers. Questions of unique factorization in integral domains are discussed and these ideas are then applied to polynomial rings and fields of rational functions. Incidentally, the characteristic of a ring is so defined that it is infinite (instead of zero) for a non-modular ring. It seems to the reviewer unfortunate to depart from common usage here, especially as the usual definition of the characteristic is just as natural as the one adopted in this book. The reviewer would also like to quarrel (in a mild way) with the author over the fact that he uses the German word "centrum" when the English word "central" has been used for this purpose for a long time.

2. One of the best examples of a non-commutative ring is the set of all square matrices of given order with coefficients in a given field. The author first studies rectangular matrices with coefficients in an arbitrary field, the results and methods being reasonably orthodox. Application is made to linear equations. Next he goes on to the study of square matrices and develops the theory of the minimum and characteristic equations, of elementary divisors, of similarity, and so on. The treatment of the minimum equation and related matters is a fine example of the elementary algebra in a non-commutative ring.

So far the results have been quite independent of the field in which the elements of the matrices lie. In the discussion of symmetric matrices (which are equivalent to quadratic forms) as well as skew, Hermitian, and other matrices, it is necessary to demand that the characteristic of the field be different from two. The author treats all these cases together in an elegant manner by introducing a general relationship that he calls an *involution* and which includes the various familiar types of symmetry as special cases. The usual questions of equivalence, canonical forms, and so on, are taken up in this general case.

In a later chapter (10) the author returns to the subject of algebras of matrices, as the most important examples of linear associative algebras; furthermore, every linear associative algebra with unit element is isomorphic to a matrix algebra. Direct sums and products of matrix algebras are discussed, as well as the characteristic and minimum equations of a matrix algebra. The important concept of scalar extension is introduced and quadrate (normal simple) algebras are briefly treated, as are the important quadrate algebras known as cyclic algebras. Finally, the algebra consisting of all polynomials in a fixed matrix with coefficients in a given field is studied.

3. As much of the theory of finite groups as is essential for the theory of Galois is presented in Chapter 6. Thus we find here a discussion of quotient groups, composition series, an introduction to groups of permutations, and so on. The theorem of primary importance to an abstract treatment of group theory is that every finite abstract group is isomorphic to a group of permutations, and this theorem is included here. However, the determination of all representations of a given abstract group by transitive groups of permutations is not given, although this is of importance in the theory of resolvents as well as in the abstract theory of groups.

In Chapter 7 the author discusses algebraic extensions. First he treats the adjunction to a fundamental field $\mathfrak{F}$ of members of a larger field which is given in advance. Then he proves the existence and uniqueness of the root field of a polynomial with coefficients in $\mathfrak{F}$. This is a field $\mathfrak{K}$ containing a subfield isomorphic to $\mathfrak{F}$ such that the given polynomial factors into linear factors in $\mathfrak{K}$; furthermore it is the "smallest" such field. The simplest example is the field of complex numbers $\mathfrak{C}$, which is the root field of x^2+1 over the field of real numbers and whose construction in elementary algebra is a sample of the construction in the general case. That $\mathfrak{C}$ itself is then the root field over $\mathfrak{C}$ of *any* polynomial with complex coefficients is, of course, the fundamental theorem of algebra. This latter is stated but not proved in this text.

This chapter also contains a discussion of separable fields and of finite fields, the so-called *Galois* fields.

In Chapter 8 the Galois theory is given, based on the study of the group of automorphisms of a normal field. The Galois group of an equation follows as a consequence. Application is then made to the classical problem of the solution of equations by radicals. It is proved that the general equation of degree greater than four is not solvable by radicals, but the existence of special equations not solvable by radicals with respect to, say, the field of rational numbers is not discussed.

In this connection the reviewer would like to see some one break with tradition and present the general theory of the reduction of the solution of an equation to auxiliary equations, instead of treating merely the question of solvability by radicals. Hölder proved his theorem on composition series in order to be able to handle the general case, and the importance of the binomial equation is perhaps more historical than actual.

A *metacyclic* field is one for which the auxiliary equations just referred to are all cyclic. The next chapter (9) in this book contributes to the discussion of this case by giving a detailed account of the structure of fields that are cyclic with respect to a given field $\mathfrak{F}$. Various cases have to be distinguished, depending on the characteristic of $\mathfrak{F}$ and the presence of certain roots of unity. Incidentally, it is found that when $\mathfrak{F}$ is modular the simplest defining equations can *not* always be taken as binomial.

4. The ideas that are grouped around the theory of valuations are of great importance in modern algebra and arithmetic. The last two chapters in this book give

a good introduction to this subject. A field $\mathfrak{F}$ has a valuation ϕ if $\phi(a)$ is a real-valued function defined for every element a of $\mathfrak{F}$ with the properties of an absolute value:

$$\phi(0)=0, \quad \phi(a)>0 \text{ if } a\neq 0, \quad \phi(ab)=\phi(a)\phi(b), \quad \phi(a+b)\leqq\phi(a)+\phi(b).$$

A field $\mathfrak{F}$ is called complete with respect to ϕ if every sequence satisfying the Cauchy criterion has a limit. If $\mathfrak{F}$ is not complete, a *derived* field can be constructed that contains a subfield $\mathfrak{F}_0$ isomorphic to $\mathfrak{F}$ and has a valuation which for elements of $\mathfrak{F}_0$ is ϕ, and which is complete with respect to this valuation. The simplest example is the field of rational numbers $\mathfrak{R}$ with ϕ the ordinary absolute value; the derived field is then the field of real numbers. The so-called p-adic valuations of $\mathfrak{R}$ lead to the corresponding p-adic derived fields that were introduced by Hensel. The latter are examples of non-archimedean valuations and these are particularly important in arithmetic.

After studying the general theory of derived fields, the author attacks the problem of determining when a field $\mathfrak{F}$ has valuations and, when it has, of finding all the possible ones. For archimedean valuations it turns out that $\mathfrak{F}$ must be equivalent to a field of complex numbers and the valuation is equivalent to the ordinary absolute value. For non-archimedean valuations the author restricts himself to algebraic number fields of finite degree over $\mathfrak{R}$ and determines them all. Thus for $\mathfrak{R}$ itself the only non-archimedean valuations are the p-adic ones already mentioned; essentially the only archimedean valuation is the ordinary absolute value.

The subjects of algebra and its close relative, arithmetic, are very much alive now, and this book is a welcome addition to the literature. Besides furnishing an excellent introduction to abstract algebra, it contains several especially valuable features, such as the chapters on matrices and matrix algebras, the chapter on cyclic fields, and the chapters on valuations.

The only thing to be criticized in the book as a whole is, in the opinion of the reviewer, the fact that the fundamental character of the concept of the ideal is not emphasized. Ideals are not introduced until Chapter 11 (they are used there in the construction of derived fields) and then their fundamental importance is not made clear. Thus the related concept of homomorphism between rings is not even mentioned. The fact that the difference ring of a commutative ring with respect to an ideal is homomorphic to the given ring and that all homomorphisms can be so obtained is of primary importance in an abstract treatment of algebra. The construction of the root field (Chapter 7), the derived field (Chapter 11), and, of course, the rings arising from congruences with respect to a fixed modulus in elementary number theory and elsewhere, are all examples of this.

The book contains a great many problems, many of them simple illustrations of the body of the text. Others are more substantial additions to the subject matter. These problems are especially numerous and valuable in the chapters on the Galois theory. At the end of the book there is a glossary in which the terms most frequently used in the book are briefly defined; this should be very useful to the beginner.

A few minor slips and misprints were noted:

Page 24, line 9 from the bottom: should be a_n, b_m instead of a, b.
Page 25, line 5 from the bottom: should be $x-c$ instead of $x-a$.
Page 76, heading at top of page should be "Chapter IV."
Page 90, line 12 from the top: should be $j\neq t$.
Page 256, line 10 from the top should read "for every non-negative integer n."
Page 279, first line should read "Hence the limit is unity as stated in our lemma."

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