THE TRANSFORMS OF FUCHSIAN GROUPS

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This paper will deal with Fuchsian linear transformations that have the unit circle

$$Q_0(z) \equiv z\bar{z} - 1 = 0$$

as principal circle. The transformations have the form

(1)
$$T(z) = \frac{az + \bar{c}}{cz + \bar{a}}, \qquad a\bar{a} - c\bar{c} = 1.$$

We shall develop several theorems concerning the relative size of the isometric circle of any transformation T of the group and of the isometric circle of

$$S(z) = GTG^{-1}(z)$$

(2)
$$= \frac{(- \alpha\bar{\alpha}a + \alpha\nu\bar{c} - \bar{\alpha}\bar{\nu}c + \nu\bar{\nu}\bar{a})z + \alpha\bar{\nu}a - \alpha^2\bar{c} + \bar{\nu}^2c - \alpha\bar{\nu}\bar{a}}{(- \bar{\alpha}\nu a + \nu^2\bar{c} - \bar{\alpha}^2c + \bar{\alpha}\nu\bar{a})z + \nu\bar{\nu}a - \alpha\nu\bar{c} + \bar{\alpha}\bar{\nu}c - \alpha\bar{\alpha}\bar{a}}$$

$$= \frac{Az + \overline{C}}{Cz + \overline{A}}, \qquad A\overline{A} - C\overline{C} = 1,$$

in which

$$G(z) = \frac{\alpha z + \bar{\nu}}{\nu z + \bar{\alpha}}, \qquad \alpha\bar{\alpha} - \nu\bar{\nu} = 1,$$

is variable and T fixed.

The concept of the isometric circle of a linear transformation was introduced in 1926 by L. R. Ford. Because of his work with it, the isometric circle now plays a fundamental part in work with groups of linear transformations. By definition* the isometric circle of a linear transformation is the complete locus of points in the neighborhood of which lengths and areas are unaltered in magnitude by the transformation. For convenience we shall sometimes use the designation $Q(z)$ for a line or circle $Q(z) = 0$.

THEOREM 1. *A necessary and sufficient condition that the radii r_s and r_t of the isometric circles of S and T be equal is that the center w of the isometric circle of the transformation G be either on the line $Q_1(z)$ through the center g_t of the isometric circle of T and the center g'_t of the isometric circle of T^{-1} or on the inverse $Q_2(z)$ of $Q_1(z)$ in $Q_0(z)$.*

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* L. R. Ford, *Automorphic Functions*, p. 25.

PROOF. The equation of Q_1 is

$$Q_1(z) = cz - \bar{c}\bar{z} - (a - \bar{a}) = 0.$$

The equation of Q_2, which is the inverse of Q_1 in Q_0, is

$$Q_2(z) = cz - \bar{c}\bar{z} - (a - \bar{a})z\bar{z} = 0.$$

If z lies on either Q_1 or Q_2, we have

(3)
$$\begin{aligned}Q_1(z)Q_2(z) = &- (a - \bar{a})(cz - \bar{c}\bar{z})(z\bar{z} + 1) + z\bar{z}[(a - \bar{a})^2 - 2c\bar{c}]\\ &+ c^2 z^2 + \bar{c}^2\bar{z}^2 = 0.\end{aligned}$$

Now, using the values of r_t and r_s from (1) and (2), we have

(4)
$$\frac{1}{r_s^2} - \frac{1}{r_t^2} = C\overline{C} - c\bar{c} = - (a - \bar{a})(- \alpha\nu\bar{c} + \bar{\alpha}\bar{\nu}c)(\alpha\bar{\alpha} + \nu\bar{\nu})$$
$$- \alpha\bar{\alpha}\nu\bar{\nu}(a - \bar{a})^2 - \alpha^2\nu^2\bar{c}^2 - \bar{\alpha}^2\bar{\nu}^2 c^2 + (\alpha^2\bar{\alpha}^2 + \nu^2\bar{\nu}^2 - 1)c\bar{c}$$

(5)
$$= - (a - \bar{a})(- \alpha\nu\bar{c} + \bar{\alpha}\bar{\nu}c)(\alpha\bar{\alpha} + \nu\bar{\nu})$$
$$- \alpha\bar{\alpha}\nu\bar{\nu}[(a - \bar{a})^2 - 2c\bar{c}] - \alpha^2\nu^2\bar{c}^2 - \bar{\alpha}^2\bar{\nu}^2 c^2,$$

which vanishes if and only if $r_s = r_t$.

Dividing (5) equated to zero by $-(\nu\bar{\nu})^2$, replacing $-\bar{\alpha}/\nu$ by w, and comparing the result with (3), we see that $r_s = r_t$ if and only if w lies on $Q_1(z)$ or $Q_2(z)$.

THEOREM 2a. *A necessary and sufficient condition that $r_s < r_t$ $(r_s > r_t)$ is that $z = w$, substituted in the expression for Q_1Q_2, make that expression negative (positive).*

PROOF. $r_s < r_t$ $(r_s > r_t)$ according as $1/r_s^2 - 1/r_t^2 > 0$ $(1/r_s^2 - 1/r_t^2 < 0)$. However,

$$Q_1Q_2 = (1/r_s^2 - 1/r_t^2)/(-\nu\bar{\nu})^2.$$

Therefore, $Q_1Q_2 < 0$ or $Q_1Q_2 > 0$ according as $1/r_s^2 - 1/r_t^2 > 0$ or $1/r_s^2 - 1/r_t^2 < 0$.

THEOREM 2b. *A necessary and sufficient condition that $r_s > r_t$ is that w be outside Q_2 and on the opposite side of Q_1 from the origin; a necessary and sufficient condition that $r_s < r_t$ is that w be inside Q_2 and on the opposite side of Q_1 from the origin or outside Q_2 and on the same side of Q_1 as the origin.[*]*

PROOF. The function Q_1 is positive (negative) according as w is on the opposite (same) side of Q_1 from (as) the origin. The function Q_2

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[*] The case of w inside Q_2 and on the same side of Q_1 as the origin is omitted since then w would be inside Q_0.

is positive (negative) according as w is outside (inside) Q_2. Comparing these statements we have the theorem:

THEOREM 3. *A necessary and sufficient condition that $r_s = r_t/k$, ($k \geq 0$), is that w lie on the locus of the equation*

$$k^2(z\bar{z} - 1)^2 = \left[z^2 - (a - \bar{a}) \frac{z}{c} - \frac{\bar{c}}{c} \right] \left[\bar{z}^2 + (a - \bar{a}) \frac{\bar{z}}{\bar{c}} - \frac{c}{\bar{c}} \right]$$

$$= \left| z - \xi_1 \right|^2 \left| z - \xi_2 \right|^2,$$

where ξ_1 and ξ_2 are the fixed points of the transformation T.

PROOF. From (4), by adding $1/r_t^2$ to each member, multiplying by $r_t^2/(\nu\bar{\nu})^2$, putting $w = -\bar{a}/\nu$, making use of the relation $a\bar{a} - \nu\bar{\nu} = 1$, putting $r_t/r_s = k$, and factoring the right-hand member, we get

(6) $k^2(w\bar{w} - 1)^2 = \left| w - \xi_1 \right|^2 \left| w - \xi_2 \right|^2,$

where

$$\xi_1, \xi_2 = \left\{ (a - \bar{a}) \pm \left[(a - \bar{a})^2 + 4c\bar{c} \right]^{1/2} \right\}/2c.$$

THEOREM 4. *If T is a hyperbolic transformation, the minimum value of k for all transformations G is given by*

$$k_0^2 = \frac{1}{4c\bar{c}} \left[(a - \bar{a})^2 + 4c\bar{c} \right] = \frac{1}{4} \left| \xi_1 - \xi_2 \right|^2;$$

this value of k is taken on if and only if w is on the circle

(7) $Q_{3,4}(z) \equiv \frac{1}{2} (a - \bar{a}) iz\bar{z} - icz + i\bar{c}\bar{z} + (a - \bar{a}) \frac{i}{2};$

for parabolic transformations zero is the lower bound for k; for elliptic transformations the minimum value of k is zero, and this value is taken on if and only if $w = \xi_1$.

PROOF. Let us consider

$$k_0^2 = \frac{1}{4c\bar{c}} \left[(a - \bar{a})^2 + 4c\bar{c} \right].$$

From (6) with the right-hand member expanded in terms of the coefficients of T, we have

$$c\bar{c}(w\bar{w} - 1)^2(k^2 - k_0^2)$$
$$= (a - \bar{a})(cw - \bar{c}\bar{w})(w\bar{w} + 1) - w\bar{w}(a - \bar{a})^2 - c^2 w^2 - \bar{c}^2 \bar{w}^2$$
$$+ (w^2 \bar{w}^2 + 1)c\bar{c} - (w\bar{w} - 1)^2(a - \bar{a})^2/4 - (w\bar{w} - 1)^2 c\bar{c}$$

$$= \left[\frac{1}{2}(a - \bar{a})iw\bar{w} - icw + i\bar{c}\bar{w} + (a - \bar{a})\frac{i}{2} \right]^2$$

$$= [Q_{3,4}(w)]^2 \geqq 0$$

for all w, since the expression in brackets is real. Hence, $k^2 = k_0^2$ if and only if w is on $Q_{3,4}$ since $|w| \neq 1$ and $c \neq 0$.

Let us notice that k^2 cannot have the value k_0^2 for elliptic transformations, since in that case $(a - \bar{a})^2 + 4cc < 0$ whereas k is a non-negative real number. Furthermore, $Q_{3,4}(w)$ has no real locus for T an elliptic transformation.

From the explicit expression for $Q_{3,4}(w)$ in the non-elliptic case it is obviously a circle with center at $-2\bar{c}/(a - \bar{a})$, a not real, and radius equal to $\{ \pm i[(a - \bar{a})^2 + 4c\bar{c}]^{1/2} \}/(a - \bar{a})$, the sign being so chosen that the radius is positive.

Let us notice that $r_{3,4} = 0$ for T parabolic, also that $Q_{3,4}(\xi) = 0$, where $\xi = (a - \bar{a})/2c$ is the fixed point of the transformation. Now for parabolic transformations we have $Q_0(\xi) = 0$; hence ξ is not a permissible value for w.

From (6) it is obvious that $k = 0$ for elliptic transformations and that $w = \xi_1, \xi_2$. However, $w \neq \xi_2$ since $Q_0(\xi_2) < 0$. We merely call attention to the fact that S is a rotation when w is equal to ξ_1, the fixed point of T which is outside Q_0.

For hyperbolic transformations ξ_1 and ξ_2 are on Q_0 and hence are not permissable values of w.

Let us now consider k_0 when a is real. From the expression given in the theorem for k_0 we see that it is unity for a real. From (6) it follows immediately that

$$(k^2 - 1)(w\bar{w} - 1)^2 c\bar{c} = -(cw - \bar{c}\bar{w})^2 \geqq 0$$

for all w since $cw - \bar{c}\bar{w}$ is a pure imaginary. Hence, for a real the minimum value of k is unity and is taken on if and only if w is on $Q_{3,4} \equiv Q_1 \equiv Q_2$. We call attention to the fact that if a is real, the transformation is hyperbolic since g_i and g_i' are on opposite ends of a diameter of Q_0.

It is readily seen from (6), with w and $\bar{w}$ as independent variables, that there are no relative maximum and minimum values of k.

REMARK. If G is so chosen that $k = k_0$, then no transform of S will increase the magnitude of its isometric circle. This is apparent since applying G_2 to $S = G_1 T G_1^{-1}$ is equivalent to applying $G_3 = G_2 G_1$ to T, but G_1 is such that I_t receives its maximum magnification.

COROLLARY $Q_{3,4}$ *is self-inverse with respect to* Q_0.

PROOF. Replacing z and $\bar{z}$ in $Q_{3,4}$ by $1/\bar{z}$ and $1/z$, respectively, and multiplying by $z\bar{z}$, we get the equation $Q_{3,4}(z) = 0$.

We found in Theorem 3 a necessary and sufficient condition that $r_s = r_t/k$ to be that w satisfy the equation.

$$k^2(z\bar{z} - 1)^2 = |z - \xi_1|^2 |z - \xi_2|^2.$$

We shall now investigate the geometric significance of that equation. Let

$$M = (a - \bar{a})^2 + 4c\bar{c},$$

$$B_3 = -[(a - \bar{a}) + (M - 4k^2c\bar{c})^{1/2}]/2\bar{c}(k^2 - 1),$$

$$B_4 = -[(a - \bar{a}) - (M - 4k^2c\bar{c})^{1/2}]/2\bar{c}(k^2 - 1),$$

$$C_3 = B_3/B_4 = [a - \bar{a} + (M - 4k^2c\bar{c})^{1/2}]^2/4c\bar{c}(k^2 - 1),$$

$$C_4 = B_4/B_3 = [a - \bar{a} - (M - 4k^2c\bar{c})^{1/2}]^2/4c\bar{c}(k^2 - 1).$$

THEOREM 5. *A necessary and sufficient condition that $r_s = r_t/k$, $(k \neq 1)$, is that w lie on either*

(8) $$Q_3(z) = z\bar{z} + B_3z + \overline{B}_3\bar{z} + C_3 = 0$$

or

(9) $$Q_4(z) = z\bar{z} + B_4z + \overline{B}_4\bar{z} + C_4 = 0.$$

PROOF. Multiplying $Q_3(z)$ by $Q_4(z)$, slightly changing the form of (6), and equating coefficients of like terms, we have the following equations that must be satisfied simultaneously:

(10) $$B_3 + B_4 = -(a - \bar{a})/\bar{c}(k^2 - 1),$$

(11) $$B_3B_4 = c/\bar{c}(k^2 - 1),$$

(12) $$B_3C_4 + B_4C_3 = -(a - \bar{a})/\bar{c}(k^2 - 1),$$

(13) $$C_3C_4 = 1,$$

(14) $$C_3 + C_4 + B_3\overline{B}_4 + \overline{B}_3B_4 = [(a - \bar{a})^2 - 2k^2c\bar{c}]/c\bar{c}(k^2 - 1).$$

Solving (10), (11), (12), and (13) we get the values stated for B_3, B_4, C_3, and C_4 and also $C_3 = C_4 = 1$. It is readily seen that the given values of B_3, B_4, and $C_3 = C_4 = 1$ do not satisfy (14). However, the given values of the four symbols do satisfy the five equations.

COROLLARY 1. *The circles Q_3 and Q_4 are always real circles for T an elliptic or a parabolic transformation; they are real circles for T a hyperbolic transformation and $k \geq k_0$, the minimum value of k for T hyperbolic.*

PROOF. Since C_3 and C_4 are real it is sufficient to show that the radius is real for each of the cases mentioned. The expression under the radical in B_3, B_4, C_3, and C_4 is readily seen to be non-positive for T an elliptic or a parabolic transformation and also for T hyperbolic provided $k \geq k_0$. Therefore,

$$r_j{}^2 = B_j\bar{B}_j - C_j, \qquad\qquad j = 3, 4,$$

$$= -\frac{[a - \bar{a} - (M - 4k^2c\bar{c})^{1/2}]^2k^2}{4c\bar{c}(k^2 - 1)^2} \geq 0.$$

COROLLARY 2. *Q_3 and Q_4 are inverse circles with respect to Q_0.*

PROOF. In order to find the inverse of a circle with respect to Q_0, we need only replace z and $\bar{z}$ by $1/\bar{z}$ and $1/z$, respectively, and multiply by $z\bar{z}$. The corollary is immediate upon performing this operation since $C_3C_4 = 1$, $B_3 = B_4/C_4$, and C_3 and C_4 are real.

COROLLARY 3. *In the hyperbolic case, the circles Q_3 and Q_4 coincide with*

$$Q_{3,4}(z) = \frac{1}{2}(a - \bar{a})iz\bar{z} - icz + ic\bar{z} + (a - \bar{a})\frac{i}{2} = 0,$$

when k is equal to k_0, the minimum value of k.

PROOF. The expression under the radical in B_3, B_4, C_3, and C_4, when evaluated for $k = k_0$, is zero; hence we have $B_3 = B_4$ and $C_3 = C_4 = 1$ and therefore $Q_3 = Q_4$. Evaluating B_3 for $k = k_0$ and putting this into the equation of Q_3, along with $C_3 = 1$, we have

$$Q_3(z) = Q_4(z) = Q_{3,4}(z).$$

COROLLARY 4. *The radius r_3 of Q_3 is equal to the product of k, the reciprocal of the magnification of r_t, and the absolute value of the center $-\bar{B}_3$ of Q_3. A similar statement is true for Q_4.*

PROOF. Symbolically, the above statement is

$$r_3{}^2 = k^2 B_3\bar{B}_3.$$

The validity of this relation is immediate upon recalling that

$$r_3{}^2 = B_3\bar{B}_3 - C_3,$$

equating the right-hand members of these two equations, and replacing B_3 and C_3 by their values as given immediately preceding theorem 5.

THEOREM 6. *Q_1, Q_2, Q_3, and Q_4 are fixed circles of the transformation T.*

PROOF. A necessary and sufficient condition that a given circle be a fixed circle of T is that it be orthogonal to both I_t and I_t'.*

Using the law of cosines, we find the angle θ between any two circles

$$q_j = z\bar{z} + b_j\bar{z} + \bar{b}_j z + c_j = 0, \qquad\qquad j = 1, 2,$$

to be given by

$$(15) \qquad\qquad \cos\theta = \frac{1}{2r_1r_2}\left[c_1 + c_2 - b_1\bar{b}_2 - \bar{b}_1b_2\right],$$

$$r_j^2 = b_j\bar{b}_j - c_j, \qquad\qquad j = 1, 2.$$

Now using the analytic expressions for Q_3 and I_t and substituting in (15), we have $\cos\theta = 0$. Hence Q_3 and I_t are orthogonal. Similarly, we find that Q_3 and I_t' are orthogonal. Therefore, the circles Q_3 are fixed circles of the transformation T. It can be shown in a similar manner that Q_1, Q_2, and Q_4 are also fixed circles of T.

THEOREM 7. *For T hyperbolic*

$$k^2 = \frac{(a - \bar{a})^2 + 4c\bar{c}}{4c\bar{c}\sin^2\theta},$$

where θ is the angle between Q_0 and Q_4.

PROOF. We shall use (15), letting $q_1(z) = Q_0(z)$ and $q_2(z) = Q_4(z)$. Then

$$\cos\theta = \frac{1}{2r_4}(-1 + C_4)$$

$$= -\frac{\left[1 + (k^2 - 1)B_4\bar{B}_4\right]}{2k(B_4\bar{B}_4)^{1/2}}$$

by Corollary 4 of Theorem 5. After putting in the value of B_4 and simplifying, we have

$$\cos\theta = -\frac{\left[(a - \bar{a})^2 - 4c\bar{c}(k^2 - 1)\right]^{1/2}}{2ik(c\bar{c})^{1/2}}.$$

Now squaring and solving for k^2, we get the value stated in the theorem.

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* L. R. Ford, *Automorphic Functions*, p. 28.